

TENSOR ALGEBRAS AND DECORATED REPRESENTATIONS

RAYMUNDO BAUTISTA AND DANIEL LÓPEZ-AGUAYO

ABSTRACT. In [2] we gave a generalization of the theory of quivers with potentials introduced by Derksen-Weyman-Zelevinsky, via completed tensor algebras over S -bimodules where S is a finite dimensional basic semisimple algebra. In this paper we show how to extend this construction to the level of decorated representations and we prove that mutation of decorated representations is an involution. Moreover, we prove that there exists a nearly Morita equivalence between the Jacobian algebras which are related via mutation. This generalizes the construction given by Buan-Iyama-Reiten-Smith in [4].

CONTENTS

1. Introduction	1
2. Preliminaries	2
3. Decorated representations	4
4. Premutation of a decorated representation	10
5. Mutation of decorated representations	37
6. Nearly Morita equivalence	47
References	53

1. INTRODUCTION

In this paper we extend the construction given in [2] to the level of decorated representations of algebras with potentials realized via completed tensor algebras. Instead of working with a quiver, we consider the algebra of formal power series $\mathcal{F}_S(M)$ where $S = \prod_{i=1}^n D_i$ is a finite direct product of division rings containing the base field F in its center and M is a finite dimensional S -bimodule. In section 3 we define a decorated representation of an algebra with potential $(\mathcal{F}_S(M), P)$ and in section 4 we show how to associate a decorated representation to the premutated algebra with potential $(\mathcal{F}_S(\mu_k M), \mu_k P)$ and we prove this is indeed a decorated representation. In contrast to [8] we do not assume that the basis is semi-multiplicative, but rather impose some conditions on the dual basis associated to the division algebra Se_k . In section 5 we define mutation of a decorated representation, and we show it is a well-defined transformation on the set of right-equivalence classes of decorated representations of algebras with potentials. A crucial result of this section is that mutation of decorated representations is an involution. Finally, in section 6 we construct a functor to prove that there exists an Nearly Morita equivalence between the Jacobian algebras which are related via mutation. This construction gives a generalization of the one given in [4].

2. PRELIMINARIES

Definition 1. Let F be a field and let D_1, \dots, D_n be division rings, containing F in its center, and each of them is finite-dimensional over F . Let $S = \prod_{i=1}^n D_i$ and let M be an S -bimodule of finite dimension over F . Define the *algebra of formal power series* over M as the set:

$$\mathcal{F}_S(M) := \left\{ \sum_{i=0}^{\infty} a(i) : a(i) \in M^{\otimes i} \right\}$$

where $M^0 = S$.

Note that $\mathcal{F}_S(M)$ is an associative unital F -algebra where the product is the one obtained by extending the product of the tensor algebra $T_S(M) = \bigoplus_{i=0}^{\infty} M^{\otimes i}$ to $\mathcal{F}_S(M)$.

Let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of S .

Definition 2. An element $m \in M$ is *legible* if $m = e_i m e_j$ for some idempotents e_i, e_j of S .

Definition 3. Let $Z = \sum_{i=1}^n F e_i \subseteq S$. We say that M is *Z-freely generated* by a Z -subbimodule M_0 of M if the map $\mu_M : S \otimes_Z M_0 \otimes_Z S \rightarrow M$ given by $\mu(s_1 \otimes m \otimes s_2) = s_1 m s_2$ is an isomorphism of S -bimodules. In this case we say that M is an S -bimodule which is *Z-free* or *Z-freely generated*.

Definition 4. Let \mathcal{C} be a non-empty subset of M . We say that \mathcal{C} is a *right S-local basis* of M if every element of \mathcal{C} is legible and if for each pair of idempotents e_i, e_j of S we have that $\mathcal{C} \cap e_i M e_j$ is a $S e_j = D_j$ -basis of $e_i M e_j$.

Definition 5. Let \mathcal{D} be a non-empty subset of M . We say that \mathcal{D} is a *left S-local basis* of M if every element of \mathcal{D} is legible and if for each pair of idempotents e_i, e_j of S we have that $\mathcal{D} \cap e_i M e_j$ is a $S e_i = D_i$ -basis of $e_i M e_j$.

A right S -local basis \mathcal{C} induces a dual basis $\{u, u^*\}_{u \in \mathcal{C}}$ of M where $u^* : M_S \rightarrow S_S$ is the morphism of right S -modules defined by $u^*(v) = 0$ if $v \in \mathcal{C} \setminus \{u\}$ and $u^*(u) = e_j$ if $u = e_i u e_j$. Similarly, a left S -local basis \mathcal{D} of M induces a dual basis $\{v, {}^*v\}_{v \in \mathcal{D}}$ where ${}^*v : {}_S M \rightarrow {}_S S$ is the morphism of left S -modules defined by ${}^*v(u) = 0$ if $u \in \mathcal{D} \setminus \{v\}$ and ${}^*v(v) = e_i$ if $v = e_i v e_j$.

Let L be a Z -local basis of S and let T be a Z -local basis of M_0 .

Throughout this paper we will use the following notation $T_k = T \cap M e_k$ and ${}_k T = T \cap e_k M$.

We will also assume that for every integer i in $[1, n]$ and for each F -basis $L(i)$ of D_i , the following equalities hold for each $f, w, z \in L(i)$:

$$(2.1) \quad f^*(w^{-1}z) = w^*(zf^{-1})$$

$$(2.2) \quad f^*(zw) = (w^{-1})^*(f^{-1}z)$$

$$(2.3) \quad z^*(wf) = (w^{-1})^*(fz^{-1})$$

Note that 2.1 readily implies (1) of [2, p.29] by taking $w = e_i$, $z = s$ and $f = t$. Observe that in 2.1 and 2.2 one can replace $z \in L(i)$ by $z \in D_i$, because both expressions are linear in z . Similarly, one can replace in 2.3 $f \in L(i)$ by $f \in D_i$.

Remark 1. If $L(i)$ is a semi-multiplicative basis of $D(i)$ then $L(i)$ satisfies 2.1, 2.2 and 2.3.

Proof. Indeed, suppose that $f^*(w^{-1}z) \neq 0$ then $w^{-1}z = cf$ for some uniquely determined $c \in F^*$; thus $f^*(w^{-1}c) = c$. On the other hand, $w^*(zf^{-1}) = w^*(z(z^{-1}wc)) = w^*(wc) = c$ and the equality follows. A similar argument shows that 2.2 and 2.3 also hold. \square

Remark 2. Suppose that L_1 is an F -basis for the field extension F_1/F and L_2 is an F_1 -basis for the field extension F_2/F_1 . If L_1 and L_2 satisfy 2.1, 2.2 and 2.3 then the F -basis $\{xy : x \in L_1, y \in L_2\}$ of F_2/F also satisfies 2.1, 2.2 and 2.3.

Proof. Suppose that both L_1 and L_2 satisfy 2.1. Let $f = f_1f_2$, $w = w_1w_2$, $z = z_1z_2$ where $f_1, w_1, z_1 \in L_1$ and $f_2, w_2, z_2 \in L_2$. Then:

$$\begin{aligned} f^*(w^{-1}z) &= f_1^*f_2^*(w_1^{-1}w_2^{-1}z_2z_1) \\ &= f_1^*(w_1^{-1}z_1f_2^*(w_2^{-1}z_2)) \\ &= f_1^*(w_1^{-1}z_1w_2^*(z_2f_2^{-1})) \\ &= w_1^*(z_1f_1^{-1}w_2^*(z_2f_2^{-1})) \\ &= (w_1w_2)^*(z_1z_2(f_1f_2)^{-1}) \\ &= w^*(zf^{-1}) \end{aligned}$$

as claimed. The equalities 2.2 and 2.3 are established in an analogous way. \square

Remark 3. If the basis $L(i)$ satisfies 2.1, then for each $f, z \in L(i)$ we have:

$$\sum_{w \in L(i)} f^*(w^{-1}z)w = zf^{-1}$$

$$\text{Proof. } \sum_{w \in L(i)} f^*(w^{-1}z)w = \sum_{w \in L(i)} w^*(zf^{-1})w = zf^{-1}.$$

Remark 4. If the basis $L(i)$ satisfies 2.2, then for each $r, v \in L(i)$ we have:

$$(2.4) \quad \sum_{t \in L(i)} r^*(vt)t^{-1} = r^{-1}v$$

$$\text{Proof. } \sum_{t \in L(i)} r^*(vt)t^{-1} = \sum_{t \in L(i)} (t^{-1})^*(r^{-1}v)t^{-1} = r^{-1}v.$$

Definition 6. Given an S -bimodule N we define the *cyclic part* of N as $N_{cyc} := \sum_{j=1}^n e_j N e_j$.

Definition 7. A *potential* P is an element of $\mathcal{F}_S(M)_{cyc}$.

For each legible element a of $e_i M e_j$, we let $\sigma(a) = i$ and $\tau(a) = j$. Recall that each $L(i) = L \cap e_i S$ is an F -basis of D_i . We will assume that each basis $L(i)$ satisfies that $\text{char}(F) \nmid \text{card } L(i)$.

Definition 8. Let P be a potential in $\mathcal{F}_S(M)$, then $R(P)$ is the closure of the two-sided ideal of $\mathcal{F}_S(M)$ generated by all the elements $X_{a^*}(P) := \sum_{s \in L(\sigma(a))} \delta_{(sa)^*}(P)s$ where $a \in T$.

In [2, p.19] it is shown that the Jacobian ideal (in the sense of [6]) contains properly the ideal $R(P)$.

Let k be an integer in $[1, n]$. Using the S -bimodule M , we define a new S -bimodule $\mu_k M = \widetilde{M}$ as:

$$\widetilde{M} := \bar{e}_k M \bar{e}_k \oplus M e_k M \oplus (e_k M)^* \oplus^* (M e_k)$$

where $\bar{e}_k = 1 - e_k$, $(e_k M)^* = \text{Hom}_S((e_k M)_S, S_S)$ and $^*(M e_k) = \text{Hom}_S(S(M e_k), S_S)$. One can show (see [2, Lemma 8.7]) that $\mu_k M$ is Z -freely generated by the following Z -subbimodule:

$$\bar{e}_k M_0 \bar{e}_k \oplus M_0 e_k S e_k M_0 \oplus e_k({}_0 N) \oplus N_0 e_k$$

where $N_0 = \{h \in M^* | h(M_0) \in Z, h(tM_0) = 0, t \in L'\}$, ${}_0 N = \{h \in {}^* M | h(M_0) \in Z, h(M_0 t) = 0, t \in L'\}$ and $L' = L \setminus \{e_1, \dots, e_n\}$.

Definition 9. An algebra with potential is a pair $(\mathcal{F}_S(M), P)$ where P is a potential in $\mathcal{F}_S(M)$ and $M_{\text{cyc}} = 0$.

Throughout this paper we will assume that M is Z -freely generated by M_0 .

3. DECORATED REPRESENTATIONS

Definition 10. Let $(\mathcal{F}_S(M), P)$ be an algebra with potential. A decorated representation of $(\mathcal{F}_S(M), P)$ is a pair $\mathcal{N} = (N, V)$ where N is a finite dimensional left $\mathcal{F}_S(M)$ -module annihilated by $R(P)$ and V is a finite dimensional left S -module.

Equivalently, N is a finite dimensional left module over the quotient algebra $\mathcal{F}_S(M)/R(P)$. For $u \in \mathcal{F}_S(M)$ we let $u_N = u : N \rightarrow N$ denote the multiplication operator $u(n) = un$.

Let $(\mathcal{F}_S(M), P)$ and $(\mathcal{F}_S(M'), P')$ be algebras with potential. Let $\mathcal{N} = (N, V)$ and $\mathcal{N}' = (N', V')$ be decorated representations of $(\mathcal{F}_S(M), P)$ and $(\mathcal{F}_S(M'), P')$, respectively. A *right-equivalence* between \mathcal{N} and \mathcal{N}' is a triple (φ, ψ, η) where:

- $\varphi : \mathcal{F}_S(M) \rightarrow \mathcal{F}_S(M')$ is a right-equivalence between $(\mathcal{F}_S(M), P)$ and $(\mathcal{F}_S(M'), P')$.
- $\psi : N \rightarrow N'$ is an isomorphism of F -vector spaces such that $\psi \circ u_N = \varphi(u)_{N'} \circ \psi$ for each $u \in \mathcal{F}_S(M)$.
- $\eta : V \rightarrow V'$ is an isomorphism of left S -modules.

Remark 5. Suppose that $M^{\otimes n} = 0$ for $n \gg 0$ and that $\mathcal{F}_S(M)_{\text{cyc}} = \{0\}$, then a decorated representation can be identified with a left module over the tensor algebra $T_S(M)$. In the case that the underlying semisimple algebra S happens to be a finite direct product of copies of the base field F , then $T_S(M)$ can be identified with a path algebra, so in this case a decorated representation is a representation of a quiver in the classical sense.

Let M_1, M_2 be Z -freely generated S -bimodules and let T_1 and T_2 be Z -free generating sets of M_1 and M_2 , respectively. Let P_1 and P_2 be potentials in $\mathcal{F}_S(M_1)$ and $\mathcal{F}_S(M_2)$ respectively, and consider the potential $P_1 + P_2 \in \mathcal{F}_S(M_1 \oplus M_2)$. Let $\mathcal{N} = (N, V)$ be a decorated representation of the algebra with potential $(\mathcal{F}_S(M_1 \oplus M_2), P_1 + P_2)$. We have an injective morphism of topological algebras $\mathcal{F}_S(M_1) \hookrightarrow \mathcal{F}_S(M_1 \oplus M_2)$ and thus, by restriction of scalars, N is a left $\mathcal{F}_S(M_1)$ -module. We will denote this module by $N|_{\mathcal{F}_S(M_1)}$. Let us show that $R(P_1)$ annihilates $N|_{\mathcal{F}_S(M_1)}$. Let $a \in T_1$, then $X_{a^*}(P_2) = \sum_{s \in L(\sigma(a))} \delta_{(sa)^*}(P_2)s = 0$. Thus $X_{a^*}(P_1 + P_2) = X_{a^*}(P_1) \in R(P_1 + P_2)$. It follows that $\mathcal{N}|_{\mathcal{F}_S(M_1)} := (N|_{\mathcal{F}_S(M_1)}, V)$ is a decorated representation of the algebra with potential $(\mathcal{F}_S(M_1), P_1)$.

Proposition 1. *Let M_1 and M_2 be Z -freely generated S -bimodules and let P, P' be reduced potentials in $\mathcal{F}_S(M_1)$ and W be a trivial potential in $\mathcal{F}_S(M_2)$. Let \mathcal{N} and \mathcal{N}' be decorated representations of $\mathcal{F}_S(M_1 \oplus M_2)$ with respect the potentials $P + W$ and $P' + W$. If \mathcal{N} is right-equivalent to \mathcal{N}' then $\mathcal{N}|_{\mathcal{F}_S(M_1)}$ is right-equivalent to $\mathcal{N}'|_{\mathcal{F}_S(M_1)}$.*

Proof. Let $(\phi, \psi, \eta) : \mathcal{N} \rightarrow \mathcal{N}'$ be a right-equivalence of decorated representations. Then

- (a) ϕ is an algebra automorphism of $\mathcal{F}_S(M_1 \oplus M_2)$ with $\phi|_S = id_S$ such that $\phi(P + W)$ is cyclically equivalent to $P' + W$.
- (b) $\psi : N \rightarrow N'$ is an isomorphism of F -vector spaces such that for $n \in N$ and $u \in \mathcal{F}_S(M_1 \oplus M_2)$ we have $\psi(un) = \phi(u)\psi(n)$.
- (c) $\eta : V \rightarrow V'$ is an isomorphism of left S -modules.

Let L be the closure of the two-sided ideal of $\mathcal{F}_S(M_1 \oplus M_2)$ generated by M_2 , then as in [2, Proposition 6.6]:

- (1) $\mathcal{F}_S(M_1 \oplus M_2) = \mathcal{F}_S(M_1) \oplus L$
- (2) $R(P + W) = R(P) \oplus L$
- (3) $R(P' + W) = R(P') \oplus L$

where $R(P)$ (respectively $R(P')$) is the closure of the two-sided ideal in $\mathcal{F}(M_1)$ generated by all the elements of the form $X_{a^*}(P)$ (respectively $X_{a^*}(P')$) where $a \in T_1$, a Z -free generating set of M_1 . Let $p : \mathcal{F}_S(M_1 \oplus M_2) \rightarrow \mathcal{F}_S(M_1)$ be the canonical projection induced by the decomposition (1). As in [2, Proposition 6.6] there exists an algebra isomorphism

$$\rho = p \circ \phi|_{\mathcal{F}_S(M_1)} : \mathcal{F}_S(M_1) \rightarrow \mathcal{F}_S(M_1)$$

such that $P' - \rho(P)$ is cyclically equivalent to an element of $R(\rho(P))^2$. By [2, Proposition 6.5] there exists an algebra automorphism λ of $\mathcal{F}_S(M_1)$ such that $\lambda\rho(P)$ is cyclically equivalent to P' and $\lambda(u) - u \in R(\rho(P))$ for all $u \in \mathcal{F}_S(M_1)$. By definition of decorated representation, $R(P + W)N = 0$ and $R(P' + W)N' = 0$. Since L is contained in both $R(P + W)$ and $R(P' + W)$ then $LN = 0$ and $LN' = 0$. Then for $u \in \mathcal{F}_S(M_1 \oplus M_2)$ and $n \in N$

$$\psi(un) = \phi(u)\psi(n)$$

Note that $\phi(u) = p\phi(u) + u'$ where $u' \in L$, then $\phi(u)\psi(n) = p\phi(u)\psi(n) = \rho(u)\psi(n)$, thus

$$\psi(un) = \rho(u)\psi(n)$$

Since $P' - \rho(P)$ is cyclically equivalent to an element of $R(\rho(P))^2$, then there exists $z \in [\mathcal{F}_S(M_1), \mathcal{F}_S(M_1)]$ such that $P' + z - \rho(P) \in R(\rho(P))^2$. Therefore by [2, Proposition 6.4] we obtain

$$R(P') = R(P' + z) = R(\rho(P))$$

Now consider the automorphism $\lambda\rho$ of $\mathcal{F}_S(M_1)$, this map has the property that $\lambda\rho(P)$ is cyclically equivalent to P' ; also for $n \in N$ and $u \in \mathcal{F}_S(M_1)$ we have

$$\psi(un) = \rho(u)\psi(n)$$

and $\lambda\rho(u) = \rho(u) + w$ where $w \in R(\rho(P)) = R(P')$. Therefore

$$\psi(un) = \lambda\rho(u)\psi(n)$$

This proves that $(\lambda\rho, \psi, \eta)$ is a right-equivalence between $\mathcal{N}|_{\mathcal{F}_S(M_1)}$ and $\mathcal{N}'|_{\mathcal{F}_S(M_1)}$, as claimed. \square

In what follows, we will use the following notation: for an S -bimodule B , define:

$$B_{\hat{k}, \hat{k}} = \bar{e}_k B \bar{e}_k$$

Now consider the algebra isomorphism $\rho : \mathcal{F}_S(M)_{\hat{k}, \hat{k}} \rightarrow \mathcal{F}_S((\mu_k M)_{\hat{k}, \hat{k}})$ defined in [2, Lemma 9.2]. Let P be a reduced potential in $\mathcal{F}_S(M)_{\hat{k}, \hat{k}}$. Suppose first that:

$$(A) \quad P = \sum_{u=1}^N f_u \gamma_u$$

where $f_u \in F$ and $\gamma_u = x_1 \dots x_{n(u)}$ with $x_i \in \hat{T}$ as in [2, Definition 26]. Let $b \in T_k$ be fixed and let N_b be the set of all $u \in [1, N]$ such that for some x_i , $a(x_i) = b$. For each $u \in N_b$, let $\mathcal{C}(u)$ be the subset of all cyclic permutations c of the set $\{1, \dots, n(u)\}$ such that $x_{c(1)} = s_c b$. Then for each $c \in \mathcal{C}(u)$ define $\gamma_u^c = x_{c(1)} x_{c(2)} \dots x_{c(n(u))}$. Thus $\gamma_u^c = s_c b r_c a_c z_c s_c$ where $z_c = x_3 \dots x_{c(n(u))}$. Therefore

$$X_{b^*}(P) = \sum_{u \in N_b} \sum_{c \in \mathcal{C}(u)} f_u r_c a_c z_c s_c$$

On the other hand:

$$\begin{aligned} X_{[bra]^*}(\rho(P)) &= \sum_{u \in N_b} \sum_{c \in \mathcal{C}(u), r_c=r, a_c=a} f_u \rho(z_c) s_c \\ &= \rho \left(\sum_{u \in N(b)} \sum_{c \in \mathcal{C}(u), r_c=r, a_c=a} f_u z_c s_c \right) \end{aligned}$$

Define $Y_{[bra]}(P) := \sum_{u \in N_b} \sum_{c \in \mathcal{C}(u), r_c=r, a_c=a} f_u z_c s_c$. Then

$$X_{[bra]^*}(\rho(P)) = \rho(Y_{[bra]}(P))$$

Note that if P is a potential in $\mathcal{F}_S(M)^{\geq n+3}$ then $Y_{[bra]}(P) \in \mathcal{F}_S(M)^{\geq n}$, thus if $(P_n)_{n \geq 1}$ is a Cauchy sequence in $\mathcal{F}_S(M)$ then $(Y_{[bra]}(P_n))_{n \geq 1}$ is Cauchy as well. Now let P be an arbitrary potential in $\mathcal{F}_S(M)$. We have

$$P = \lim_{n \rightarrow \infty} P_n$$

where each P_n is of the form given by (A). Define:

$$w = \lim_{n \rightarrow \infty} Y_{[bra]}(P_n)$$

Then

$$\begin{aligned} X_{[bra]^*}(\rho(P)) &= \lim_{n \rightarrow \infty} X_{[bra]^*}(\rho(P_n)) \\ &= \lim_{n \rightarrow \infty} \rho(Y_{[bra]}(P_n)) \\ &= \rho(\lim_{n \rightarrow \infty} Y_{[bra]}(P_n)) \\ &= \rho(w) \end{aligned}$$

Thus we let $Y_{[bra]}(P) := w$. Then $X_{[bra]^*}(\rho(P)) = \rho(Y_{[bra]}(P))$. In [2, p.60] the following equalities are established for each potential P in $T_S(M)$:

$$(3.1) \quad \rho(b' X_{b^*}(P)) = \sum_{r \in L(k), a \in {}_k T} [b' r a] X_{[bra]^*}(\rho(P))$$

$$(3.2) \quad \rho(X_{a^*}(P) a') = \sum_{b \in T_k, r \in L(k)} X_{[bra]^*}(\rho(P)) [bra']$$

By continuity, the above formulas remain valid for every potential $P \in \mathcal{F}_S(M)$. Using 3.1 yields:

$$\begin{aligned}\rho(b'X_{b^*}(P)) &= \sum_{r \in L(k), a \in {}_k T} \rho(b'ra)\rho(Y_{[bra]}(P)) \\ &= \rho\left(\sum_{r \in L(k), a \in {}_k T} b'raY_{[bra]}(P)\right)\end{aligned}$$

Since ρ is injective, then:

$$(3.3) \quad b'X_{b^*}(P) = \sum_{r \in L(k), a \in {}_k T} b'raY_{[bra]}(P)$$

Similarly:

$$(3.4) \quad X_{a^*}(P)a' = \sum_{b \in T_k, r \in L(k)} Y_{[bra]}(P)bra'$$

For each $\psi \in M^*$ and for each positive integer n we have an F -linear map $\psi_* : M^{\otimes n} \rightarrow M^{\otimes(n-1)}$ given by $\psi_*(m_1 \otimes m_2 \otimes \dots \otimes m_n) = \psi(m_1)m_2 \otimes \dots \otimes m_n$. This map induces an F -linear map $\psi_* : \mathcal{F}_S(M) \rightarrow \mathcal{F}_S(M)$. Similarly, if $\eta \in {}^*M$ then we obtain an F -linear map ${}^*\eta : M^{\otimes n} \rightarrow M^{\otimes(n-1)}$ given by ${}^*\eta(m_1 \otimes \dots \otimes m_{n-1} \otimes m_n) = m_1 \otimes \dots \otimes m_{n-1}\eta(m_n)$. Now suppose that $b' \in T_k$, then $b'e_k = b'$ and thus $e_k(b')^* = (b')^*$. Since X^P is a morphism of S -bimodules [2, Proposition 7.6] then $e_kX_{b^*}(P) = X_{b^*}(P)$. Applying the map $(b')^* : \mathcal{F}_S(M) \rightarrow \mathcal{F}_S(M)$ to the left-hand side of 3.3 yields

$$(3.5) \quad X_{b^*}(P) = e_kX_{b^*}(P) = \sum_{r \in L(k), a \in {}_k T} e_kraY_{[bra]}(P)$$

Therefore

$$(3.6) \quad X_{b^*}(P) = \sum_{r \in L(k), a \in {}_k T} raY_{[bra]}(P)$$

Let H denote the set of all non-zero elements of the form as where $a \in {}_k T$ and $s \in L$. Note that H is a left S -local basis of ${}_S M$; for $x \in H$ we denote by ${}^*x \in {}^*M$ the map given by ${}^*x(y) = 0$ if $y \in H \setminus \{x\}$ and $x^*(x) = e_i$ where $e_i x = x$. Applying the map ${}^*(a')$ to 3.4 we obtain:

$$(3.7) \quad X_{a^*}(P) = \sum_{b \in T_k, r \in L(k)} Y_{[bra]}(P)br$$

Let $\mathcal{N} = (N, V)$ be a decorated representation of the algebra with potential $(\mathcal{F}_S(M), P)$ and suppose that k satisfies $(Me_k \otimes_S M)_{cyc} = \{0\}$. Define:

$$N_{in} = \bigoplus_{a \in {}_k T} D_k \otimes_F N_{\tau(a)}$$

$$N_{out} = \bigoplus_{b \in T_k} D_k \otimes_F N_{\sigma(b)}$$

For each a in ${}_k T$ and $r \in L(k)$ consider the projection map $\pi'_a : N_{in} \rightarrow D_k \otimes_F N_{\tau(a)}$ and the map $\pi'_{ra} : D_k \otimes_F N_{\tau(a)} \rightarrow N_{\tau(a)}$ given by $\pi'_{ra}(d \otimes n) = r^*(d)n$. Let $\xi'_a : D_k \otimes_F N_{\tau(a)} \rightarrow N_{in}$ denote

the inclusion map and define $\xi'_{ra} : N_{\tau(a)} \rightarrow D_k \otimes_F N_{\tau(a)}$ as the map given by $\xi'_{ra}(n) = r \otimes n$.

Then for $r, r_1 \in L(k)$ we have the following equalities

$$(3.8) \quad \pi'_{r_1 a} \xi'_{ra} = \delta_{r, r_1} id_{N_{\tau(a)}}$$

$$(3.9) \quad \pi'_{ra} \xi'_{ra} = id_{N_{\tau(a)}}$$

For $a \in_k T$ and $r \in L(k)$ we define the following F -linear maps:

$$\begin{aligned} \pi_{ra} &= \pi'_{ra} \pi'_a : N_{in} \rightarrow N_{\tau(a)} \\ \xi_{ra} &= \xi'_a \xi'_{ra} : N_{\tau(a)} \rightarrow N_{in} \end{aligned}$$

then we have the following equalities

$$(3.10) \quad \pi_{r_1 a_1} \xi_{ra} = \delta_{r_1 a_1, ra} id_{N_{\tau(a)}}$$

$$(3.11) \quad \sum_{r \in L(k), a \in_k T} \xi_{ra} \pi_{ra} = id_{N_{in}}$$

Similarly, for each $r \in L(k)$ and $b \in T_k$ we have the canonical projection $\pi'_b : N_{out} \rightarrow D_k \otimes_F N_{\sigma(b)}$ and $\pi'_{br} : D_k \otimes_F N_{\sigma(b)} \rightarrow N_{\sigma(b)}$ denotes the map given by $\pi'_{br}(d \otimes n) = (r^{-1})^*(d)n$.

We define $\xi'_{br} : N_{\sigma(b)} \rightarrow D_k \otimes_F N_{\sigma(b)}$ as the map given by $\xi'_{br}(n) = r^{-1} \otimes n$ for every $n \in N_{\sigma(b)}$ and $\xi'_b : D_k \otimes_F N_{\sigma(b)} \rightarrow N_{out}$ is the inclusion map.

Then for $r, r_1 \in L(k)$ and $b \in T_k$ we have the following equalities:

$$(3.12) \quad \pi'_{br_1} \xi'_{br} = \delta_{r_1, r} id_{N_{\sigma(b)}}$$

$$(3.13) \quad \pi'_{br} \xi'_{br} = id_{N_{\sigma(b)}}$$

Define the following F -linear maps:

$$\begin{aligned} \xi_{br} &= \xi'_b \xi'_{br} : N_{\sigma(b)} \rightarrow N_{out} \\ \pi_{br} &= \pi'_{br} \pi'_b : N_{out} \rightarrow N_{\sigma(b)} \end{aligned}$$

Then for $r, r_1 \in L(k)$ and $b, b_1 \in T_k$ we have

$$(3.14) \quad \pi_{b_1 r_1} \xi_{br} = \delta_{b_1 r_1, br} id_{N_{\sigma(b)}}$$

and

$$(3.15) \quad \sum_{b \in T_k, r \in L(k)} \xi_{br} \pi_{br} = id_{N_{out}}$$

We define a map of left D_k -modules $\alpha : N_{in} \rightarrow N_k$ as the map such that for all $a \in_k T, r \in L(k)$ we have:

$$\alpha \xi_{ra}(n) = ran$$

for each $n \in N_{\tau(a)}$.

Similarly, we define $\beta : N_k \rightarrow N_{out}$ as the F -linear map such that for all $b \in T_k, r \in L(k)$:

$$\pi_{br}\beta(n) = brn$$

for every $n \in N_k$.

Finally, the map $\gamma : N_{out} \rightarrow N_{in}$ is the morphism of left D_k -modules such that map $\gamma_{ra,bs} = \pi_{ra}\gamma\xi_{bs} : N_{\sigma(b)} \rightarrow N_{\tau(a)}$ where $r, s \in L(k)$, $a \in_k T$, $b \in T_k$, is given by:

$$\gamma_{ra,bs}(n) = \sum_{w \in L(k)} r^*(s^{-1}w)Y_{[bwa]}(P)n$$

for every $n \in N_{\sigma(b)}$.

Proposition 2. *The map β is a morphism of left D_k -modules.*

Proof. By linearity, it suffices to show that if $c \in L(k)$ and $n \in N_k$ then $\beta(cn) = c\beta(n)$. Using [2, Proposition 7.5] we obtain:

$$\begin{aligned} \beta(cn) &= c \sum_{b \in T_k, r \in L(k)} c^{-1}\xi_{br}\pi_{br}\beta(cn) = c \sum_{b \in T_k, r \in L(k)} c^{-1}(r^{-1} \otimes brcn) \\ &= c \sum_{b \in T_k, r, r_1, r_2 \in L(k)} (r_1^{-1})^*(c^{-1}r^{-1})r_1^{-1} \otimes br_2^*(rc)r_2n \\ &= c \sum_{b \in T_k, r_1, r_2 \in L(k)} \left(\sum_{r \in L(k)} r_2^*(rc)(r_1^{-1})^*(c^{-1}r^{-1}) \right) (r_1^{-1} \otimes br_2n) \\ &= c \left(\sum_{b \in T_k, r \in L(k)} r^{-1} \otimes brn \right) \\ &= c\beta(n) \end{aligned}$$

as claimed. □

Lemma 1. *We have $\alpha\gamma = 0$ and $\gamma\beta = 0$.*

Proof. We first show that $\alpha\gamma = 0$. It suffices to show that for all $r \in L(k)$, $b \in T_k$, $\alpha\gamma\xi_{br} = 0$. Let $n \in N_{\sigma(b)}$, then by 3.6 and 3.11:

$$\begin{aligned} \alpha\gamma\xi_{br}(n) &= \alpha id_{N_{in}}\gamma\xi_{br}(n) = \sum_{s \in L(k), a \in_k T} \alpha\xi_{sa}\pi_{sa}\gamma\xi_{br}(n) \\ &= \sum_{s \in L(k), a \in_k T} \alpha\xi_{sa}\gamma_{sa,br}(n) \\ &= \sum_{s, w \in L(k), a \in_k T} \alpha\xi_{sa}s^*(r^{-1}w)Y_{[bwa]}(P)(n) \\ &= \sum_{s, w \in L(k), a \in_k T} sas^*(r^{-1}w)Y_{[bwa]}(P)(n) \\ &= \sum_{w \in L(k), a \in_k T} r^{-1}waY_{[bwa]}(P)n \\ &= r^{-1}X_{b^*}(P)n \\ &= 0 \end{aligned}$$

We now show that $\gamma\beta = 0$. It suffices to show that for all $r \in L(k)$, $a \in_k T$ we have $\pi_{ra}\gamma\beta = 0$. Let $n \in N_k$, then by 3.7 and 3.15:

$$\begin{aligned}
\pi_{ra}\gamma\beta(n) &= \pi_{ra}\gamma id_{N_{out}}\beta = \sum_{b \in T_k, s \in L(k)} \pi_{ra}\gamma\xi_{bs}\pi_{bs}\beta(n) \\
&= \sum_{b \in T_k, s \in L(k)} \gamma_{ra,bs}\pi_{bs}\beta(n) \\
&\quad \sum_{b \in T_k, s, w \in L(k)} s^*(wr^{-1})Y_{[bwa]}(P)(bsn) \\
&= \sum_{b \in T_k, w \in L(k)} Y_{[bwa]}(P)bwr^{-1}n \\
&= X_{a^*}(P)r^{-1}n \\
&= 0
\end{aligned}$$

□

Lemma 2. For each $m \in N_{in}$, $a \in_k T$ we have $\pi_{eka}(r^{-1}m) = \pi_{ra}(m)$.

Proof. First, for any $a_1 \in_k T$ and $n \in N_{\tau(a_1)}$ we have

$$r^{-1}\xi'_{sa_1}(n) = r^{-1}(s \otimes n) = r^{-1}s \otimes n = \sum_{u \in L(k)} u \otimes u^*(r^{-1}s)n = \sum_{u \in L(k)} \xi'_{ua_1}(u^*(r^{-1}s)n)$$

Then

$$r^{-1}\xi_{sa_1}(n) = r^{-1}\xi'_{a_1}\xi'_{sa_1}(n) = \sum_{u \in L(k)} \xi'_{a_1}\xi'_{ua_1}(u^*(r^{-1}s)n) = \sum_{u \in L(k)} \xi_{ua_1}(u^*(r^{-1}s)n)$$

Now let $m \in N_{in}$, then using 3.10 and 3.11 we obtain

$$\begin{aligned}
\pi_{eka}(r^{-1}m) &= \sum_{s \in L(k), a_1 \in_k T} \pi_{eka}r^{-1}\xi_{sa_1}\pi_{sa_1}(m) \\
&= \sum_{s, u \in L(k), a_1 \in_k T} \pi_{eka}\xi_{ua_1}(u^*(r^{-1}s)\pi_{sa_1}(m)) \\
&= \sum_{s \in L(k)} e_k^*(r^{-1}s)\pi_{sa}(m) \\
&= \pi_{ra}(m)
\end{aligned}$$

and the lemma follows. □

4. PERMUTATION OF A DECORATED REPRESENTATION

Consider now the algebra with potential $(\mathcal{F}_S(\widetilde{M}), \widetilde{P})$. Recall from [2, Definition 37] that:

$$\begin{aligned}
\widetilde{M} &:= \bar{e}_k M \bar{e}_k \oplus M e_k M \oplus (e_k M)^* \oplus^* (M e_k) \\
\widetilde{P} &:= [P] + \sum_{sa \in_k \hat{T}, bt \in \hat{T}_k} [btsa]((sa)^*)(^*(bt))
\end{aligned}$$

To a decorated representation $\mathcal{N} = (N, V)$ of the algebra with potential $(\mathcal{F}_S(M), P)$ we will associate a decorated representation $\widetilde{\mu}_k(\mathcal{N}) = (\overline{N}, \overline{V})$ of $(\mathcal{F}_S(\widetilde{M}), \widetilde{P})$ as follows. First set:

$$\overline{N}_i = N_i, \overline{V}_i = V_i \text{ if } i \neq k$$

Define \overline{N}_k and \overline{V}_k as follows:

$$\begin{aligned}\overline{N}_k &= \frac{\ker(\gamma)}{\operatorname{im}(\beta)} \oplus \operatorname{im}(\gamma) \oplus \frac{\ker(\alpha)}{\operatorname{im}(\gamma)} \oplus V_k \\ \overline{V}_k &= \frac{\ker(\beta)}{\ker(\beta) \cap \operatorname{im}(\alpha)}\end{aligned}$$

Let

$$(4.1) \quad \begin{aligned}J_1 &: \frac{\ker(\gamma)}{\operatorname{im}(\beta)} \rightarrow \overline{N}_k \\ J_2 &: \operatorname{im}(\gamma) \rightarrow \overline{N}_k \\ J_3 &: \frac{\ker(\alpha)}{\operatorname{im}(\gamma)} \rightarrow \overline{N}_k \\ J_4 &: V_k \rightarrow \overline{N}_k\end{aligned}$$

be the corresponding inclusions and let

$$(4.2) \quad \begin{aligned}\Pi_1 &: \overline{N}_k \rightarrow \frac{\ker(\gamma)}{\operatorname{im}(\beta)} \\ \Pi_2 &: \overline{N}_k \rightarrow \operatorname{im}(\gamma) \\ \Pi_3 &: \overline{N}_k \rightarrow \frac{\ker(\alpha)}{\operatorname{im}(\gamma)} \\ \Pi_4 &: \overline{N}_k \rightarrow V_k\end{aligned}$$

denote the canonical projections.

Remark. Suppose that M is Z -freely generated by M_0 and let X be a finite dimensional left S -module. To induce a structure of a $T_S(M)$ -left module on X it suffices to give a map of S -left modules $M \otimes_S X \rightarrow X$. Let $i \neq j$ be integers in $[1, n]$. Then

$$\begin{aligned}\operatorname{Hom}_{D_i}(e_i M e_j \otimes_S X, X) &\cong \operatorname{Hom}_{D_i}((D_i \otimes_F e_i M_0 e_j \otimes_F D_j) \otimes_{D_j} e_j X, e_i X) \\ &\cong \operatorname{Hom}_{D_i}(D_i \otimes_F e_i M_0 e_j \otimes_F (D_j \otimes_{D_j} e_j X), e_i X) \\ &\cong \operatorname{Hom}_{D_i}(D_i \otimes_F e_i M_0 e_j \otimes_F e_j X, e_i X) \\ &\cong \operatorname{Hom}_F(e_i M_0 e_j \otimes_F e_j X, e_i X)\end{aligned}$$

Hence $\operatorname{Hom}_S({}_S(M \otimes_S X), {}_S X) \cong \bigoplus_{i,j} \operatorname{Hom}_F(e_i M_0 e_j \otimes_F e_j X, e_i X)$ as F -vector spaces. Therefore, a map of left S -modules $M \otimes_S X \rightarrow X$ is determined by a collection of F -linear maps $\theta_{i,j} : e_i M_0 e_j \otimes_F e_j X \rightarrow e_i X$. It follows that each element $c \in e_{\sigma(c)} M_0 e_{\tau(c)}$ gives rise to a multiplication operator $c_X : X_{\tau(c)} \rightarrow X_{\sigma(c)}$ given by $c_X(x) := \theta_{\sigma(c), \tau(c)}(c \otimes x)$.

Recall from [2, Lemma 8.7] that \widetilde{M} is Z -freely generated by the following Z -subbimodule:

$$(\widetilde{M})_0 := \bar{e}_k M_0 \bar{e}_k \oplus M_0 e_k S e_k M_0 \oplus e_k({}_0 N) \oplus N_0 e_k$$

To give \overline{N} a structure of a left $T_S(\widetilde{M})$ -module on \overline{N} we will proceed by cases, by giving the action of each summand of $(\widetilde{M})_0$ in \overline{N} .

- Suppose first that $i, j \neq k$. Then

$$e_i(\widetilde{M})_0 e_j = e_i M_0 e_j \oplus e_i M_0 e_k S e_k M_0 e_j$$

Assume that $c \in e_i M_0 e_j$. By assumption $i, j \neq k$ and thus both $\sigma(c)$ and $\tau(c)$ are not equal to k . Then $\overline{N}_{\tau(c)} = N_{\tau(c)}$ and $\overline{N}_{\sigma(c)} = N_{\sigma(c)}$. Therefore we set $c_{\overline{N}} = c_N$. Assume now that c is an element of the Z -local basis of $e_i M_0 e_k S e_k M_0 e_j$ then $c = \rho(bra)$ for some $b \in T \cap e_i M e_k$, $r \in L(k)$ and $a \in T \cap e_k M e_j$. In this case we set $\rho(bra)_{\overline{N}} := (bra)_N$.

Recall that $\{^*b : b \in T\}$ is a Z -local basis of ${}_0N$ and $\{a^* : a \in T\}$ is a Z -local basis of N_0 . Then $\{^*b : b \in T_k\}$ is a Z -free generating set of $e_k(^*M)$ and $\{a^* : a \in {}_kT\}$ is a Z -free generating set of M^*e_k . Suppose that $b = e_{\sigma(b)}be_k$, then $\tau(^*b) = \sigma(b)$. Therefore

$$\begin{aligned} \overline{N}_{in} &= \bigoplus_{b \in T_k} D_k \otimes_F N_{\tau(^*b)} \\ &= \bigoplus_{b \in T_k} D_k \otimes_F N_{\sigma(b)} \\ &= N_{out} \end{aligned}$$

whence $\overline{N}_{in} = N_{out}$. A similar argument shows that $\overline{N}_{out} = N_{in}$.

We have the inclusion maps

$$\begin{aligned} j : \ker(\gamma) &\rightarrow N_{out} \\ i : \text{im}(\gamma) &\rightarrow N_{in} \\ j' : \ker(\alpha) &\rightarrow N_{in} \end{aligned}$$

and the canonical projections

$$\begin{aligned} \pi_1 : \ker(\gamma) &\rightarrow \frac{\ker(\gamma)}{\text{im}(\beta)} \\ \pi_2 : \ker(\alpha) &\rightarrow \frac{\ker(\alpha)}{\text{im}(\gamma)} \end{aligned}$$

As in [5] we introduce the following splitting data:

- Choose a D_k -linear map $p : N_{out} \rightarrow \ker(\gamma)$ such that $pj = id_{\ker(\gamma)}$.
- Choose a D_k -linear map $\sigma_2 : \ker(\alpha)/\text{im}(\gamma) \rightarrow \ker(\alpha)$ such that $\pi_2\sigma_2 = id_{\ker(\alpha)/\text{im}(\gamma)}$.

• Suppose now that $i \neq k$ and that $j = k$. Then $e_i(\widetilde{M})_0 e_k = e_i(N_0)e_k$. Let $a \in {}_kT$, then $\tau(a^*) = k$ and $\sigma(a^*) = \tau(a)$. We define an F -linear map:

$$\overline{N}(a^*) : \overline{N}_k \rightarrow N_{\tau(a)}$$

as follows

$$\begin{aligned} \overline{N}(a^*)J_1 &= 0 \\ \overline{N}(a^*)J_2 &= c_k^{-1}\pi_{e_ka}i \\ \overline{N}(a^*)J_3 &= c_k^{-1}\pi_{e_ka}j'\sigma_2 \\ \overline{N}(a^*)J_4 &= 0 \end{aligned} \tag{4.3}$$

where $c_k = [D_k : F]$.

• In what follows, we let $\gamma = i\gamma'$ where $\gamma' : N_{out} \twoheadrightarrow im(\gamma)$. Suppose now that $i = k$ and that $j \neq k$. Then

$$e_i(\widetilde{M})_0 e_j = e_k(0N) e_j$$

Since $j \neq k$ then $\overline{N}_j = N_j = e_j N$. For every $b \in T_k$, we define an F -linear map:

$$\overline{N}(*b) : N_{\sigma(b)} \rightarrow \overline{N}_k$$

as follows

$$(4.4) \quad \begin{aligned} \Pi_1 \overline{N}(*b) &= -\pi_1 p \xi_{be_k} \\ \Pi_2 \overline{N}(*b) &= -\gamma' \xi_{be_k} \\ \Pi_3 \overline{N}(*b) &= 0 \\ \Pi_4 \overline{N}(*b) &= 0 \end{aligned}$$

The previous construction makes \overline{N} a left $T_S(\widetilde{M})$ -module. To see that \overline{N} is in fact a module over the completed algebra $\mathcal{F}_S(\widetilde{M})$ it suffices to note that the $\mathcal{F}_S(M)$ -module N is nilpotent [5, p. 39] and thus \overline{N} is annihilated by $\langle \widetilde{M} \rangle^n$ for large enough n .

Lemma 3. *Let $\rho : \mathcal{F}_S(M)_{\hat{k}, \hat{k}} \rightarrow \mathcal{F}_S((\mu_k M)_{\hat{k}, \hat{k}})$ be the algebra isomorphism introduced on page 5 and let $u \in \mathcal{F}_S(M)_{\hat{k}, \hat{k}}$. Then $\rho(u)_{\overline{N}} = u_N$.*

Proof. First note that $\mathcal{F}_S(M) = S \oplus M \oplus \mathcal{F}_S(M)^{\geq 2}$. Let $u \in \mathcal{F}_S(M)_{\hat{k}, \hat{k}}$, then $u = s + m + x$ where $s \in \bar{e}_k S$, $m \in M_{\hat{k}, \hat{k}}$ and $x \in (\mathcal{F}_S(M)^{\geq 2})_{\hat{k}, \hat{k}}$. Then

$$\rho(u) = s + m + \rho(x)$$

By continuity and linearity of ρ , it suffices to treat the case when x is of the form $s(x_1)x_1s(x_2)\dots s(x_l)x_l$, where $s(x_i) \in L(\sigma(x_i))$ and $x_i \in T$. We'll use induction on l . Suppose first that $x = s(x_1)x_1s(x_2)x_2$ and we may assume that $x_1s(x_2)x_2 \in M_0e_kSe_kM_0$, then

$$\rho(x) = s(x_1)\rho(x_1s(x_2)x_2)$$

Therefore

$$\rho(x)n = s(x_1)\rho(x_1s(x_2)x_2)n$$

Since $[b_qra_s]_{\overline{N}} = (b_qra_s)_N$ then $\rho(b_qra_s)n = b_qra_sn$. It follows that

$$\begin{aligned} \rho(x)n &= s(x_1)\rho(x_1s(x_2)x_2)n \\ &= s(x_1)x_1s(x_2)x_2n \\ &= xn \end{aligned}$$

Suppose now that the claim holds for the length of x less than n . We have:

$$x = s(x_1)x_1s(x_2)x_2 \dots s(x_{l-2})x_{l-2}s(x_{l-1})x_{l-1}s(x_l)x_l$$

Using the fact that ρ is an algebra morphism together with the base case $l = 2$ we obtain:

$$\begin{aligned}\rho(x)n &= \rho(s(x_1)x_1 \dots s(x_{l-2})x_{l-2})\rho(s(x_{l-1})x_{l-1}s(x_l)x_l)n \\ &= \rho(s(x_1)x_1 \dots (x_{l-2})x_{l-2})s(x_{l-1})\rho(x_{l-1}s(x_l)x_l)n \\ &= \rho(s(x_1)x_1 \dots s(x_{l-2})x_{l-2})s(x_{l-1})x_{l-1}s(x_l)x_l n \\ &= \rho(s(x_1)x_1 \dots s(x_{l-2})x_{l-2})n'\end{aligned}$$

where $n' := s(x_{l-1})x_{l-1}s(x_l)x_l n$. Since $s(x_1)x_1 \dots s(x_{l-2})x_{l-2}$ has length less than l , then:

$$\rho(s(x_1)x_1 \dots s(x_{l-2})x_{l-2})n' = s(x_1)x_1 \dots s(x_{l-2})x_{l-2}n'$$

Therefore

$$\begin{aligned}\rho(x)n &= s(x_1)x_1 \dots s(x_{l-2})x_{l-2}n' \\ &= s(x_1)x_1 \dots s(x_{l-2})x_{l-2}s(x_{l-1})x_{l-1}s(x_l)x_l n \\ &= xn\end{aligned}$$

it follows that $\rho(x)n = xn$ completing the proof. \square

Proposition 3. *The pair $\widetilde{\mu}_k(\mathcal{N}) = (\overline{N}, \overline{V})$ is a decorated representation of $(\mathcal{F}_S(\widetilde{M}), \widetilde{P})$.*

Proof. We have to verify that \overline{N} is annihilated by $R(\widetilde{P})$. It suffices to check that $(X_{c^*}(\widetilde{P}))_{\overline{N}} = 0$ for each element c of the Z -local basis of $(\widetilde{M})_0$. We proceed by cases.

- Suppose first that $c \in T \cap \bar{e}_k M_0 \bar{e}_k$ and let $n \in N$. Then by Lemma 3

$$\begin{aligned}\left(X_{c^*}(\widetilde{P})_{\overline{N}}\right)(n) &= X_{c^*}(\widetilde{P})n \\ &= X_{c^*}(\rho(P))n \\ &= \rho(X_{c^*}(P))n \\ &= X_{c^*}(P)n\end{aligned}$$

Since $\mathcal{N} = (N, V)$ is a decorated representation of $(\mathcal{F}_S(M), P)$ then $X_{c^*}(P)n = 0$.

- Suppose now that $c = \rho(bra)$ where $b \in T_k$, $r \in L(k)$ and $a \in_k T$. By [2, p.58] we have the following equality:

$$X_{[bra]^*}(\widetilde{P}) = X_{[bra]^*}(\rho(P)) + c_k a^* r^{-1}(*b)$$

where $c_k = [D_k : F]$.

We now compute the image of the operator $(c_k a^* r^{-1}(*b))_{\overline{N}}$. Let $n \in N_{\sigma(b)}$, then remembering 4.3, 4.4 and Lemma 2 we obtain

$$\begin{aligned}
 c_k \overline{N}(a^*) r^{-1} \overline{N}(*b)(n) &= c_k \overline{N}(a^*) r^{-1} (-\pi_1 p \xi_{be_k}(n), -\gamma' \xi_{be_k}(n), 0, 0) \\
 &= c_k \overline{N}(a^*) \left(-r^{-1} \pi_1 p \xi_{be_k}(n), -r^{-1} \gamma' \xi_{be_k}(n), 0, 0 \right) \\
 &= c_k c_k^{-1} \pi_{e_k a} i \left(-r^{-1} \gamma' \xi_{be_k}(n) \right) \\
 &= -\pi_{e_k a} \left(r^{-1} \gamma' \xi_{be_k}(n) \right) \\
 &= -\pi_{ra} (\gamma' \xi_{be_k}(n)) \\
 &= -\gamma_{ra, be_k}(n) \\
 &= - \sum_{w \in L(k)} r^*(e_k^{-1} w) Y_{[bwa]}(P) n \\
 &= - \sum_{w \in L(k)} r^*(w) Y_{[bwa]}(P) n \\
 &= -Y_{[bra]}(P) n
 \end{aligned}$$

and by Lemma 3

$$\begin{aligned}
 (X_{[bra]^*}(\rho(P)))_{\overline{N}}(n) &= X_{[bra]^*}(\rho(P))n \\
 &= \rho(Y_{[bra]}(P))n \\
 &= Y_{[bra]}(P)n
 \end{aligned}$$

Combining the above: $(X_{[bra]}(\tilde{P}))_{\overline{N}} = (X_{[bra]^*}(\rho(P)))_{\overline{N}} + (c_k a^* r^{-1}(*b))_{\overline{N}} = 0$, as desired. It remains to show that $R(\tilde{P}) \cdot \overline{N} = \{0\}$ for the remaining elements of the Z -local basis of $(\tilde{M})_0$. We now show that $(X_{a^*}(\tilde{P}))_{\overline{N}} = 0$ for each $a \in_k T$. Using the result of page [2, p.58] we have that

$$\begin{aligned}
 (X_{a^*}(\tilde{P}))_{\overline{N}} &= \left(c_k \sum_{b \in T_k, r \in L(k)} r^{-1}(*b) \rho(bra) \right)_{\overline{N}} \\
 &= c_k \sum_{b \in T_k, r \in L(k)} (r^{-1}(*b) \rho(bra))_{\overline{N}}
 \end{aligned}$$

Let $n \in N_{\tau(a)}$. Then remembering 4.4 and 3.15, we get the following equalities

$$\begin{aligned}
 X_{a^*}(\tilde{P})(n) &= c_k \sum_{b \in T_k, r \in L(k)} r^{-1}(*b)(bran) \\
 &= c_k \sum_{b \in T_k, r \in L(k)} r^{-1} (-\pi_1 p \xi_{be_k}(bran), -\gamma' \xi_{be_k}(bran), 0, 0) \\
 &= -c_k \left(\sum_{b \in T_k, r \in L(k)} r^{-1} \pi_1 p \xi_{be_k}(bran), \sum_{b \in T_k, r \in L(k)} r^{-1} \gamma' \xi_{be_k}(bran), 0, 0 \right) \\
 &= -c_k \left(\sum_{b \in T_k, r \in L(k)} \pi_1 p r^{-1} \xi_{be_k}(bran), \sum_{b \in T_k, r \in L(k)} \gamma' r^{-1} \xi_{be_k}(bran), 0, 0 \right) \\
 &= -c_k \left(\sum_{b \in T_k, r \in L(k)} \pi_1 p \xi_{br} \pi_{br} \beta(an), \sum_{b \in T_k, r \in L(k)} \gamma' \xi_{br} \pi_{br} \beta(an), 0, 0 \right) \\
 &= -c_k (\pi_1 \beta(an), \gamma' \beta(an), 0, 0) \\
 &= (0, 0, 0, 0)
 \end{aligned}$$

by Lemma 1. This proves that $(X_{a*}(\tilde{P}))_{\overline{N}} = 0$ for each $a \in_k T$. Finally, let us show that $(X_{*b}(\tilde{P}))_{\overline{N}} = 0$ for each $b \in T_k$. Let us recall the following formula from [2, p.58]:

$$X_{*(b)}(\tilde{P}) = c_k \sum_{a \in_k T, r \in L(k)} \rho(bra) a^* r^{-1}$$

Now let $n \in N_k$. Then remembering 4.3 and using Lemma 2 we get

$$\begin{aligned} X_{*b}(\tilde{P}) &= c_k \sum_{a \in_k T, r \in L(k)} \rho(bra) \overline{N}(a^*)(r^{-1}n) \\ &= c_k \sum_{a \in_k T, r \in L(k)} \rho(bra) \overline{N}(a^*) \left(\sum_{l=1}^4 J_l \Pi_l(r^{-1}n) \right) \\ &= c_k \sum_{a \in_k T, r \in L(k)} \left(\rho(bra) c_k^{-1} \pi_{e_k a} i \Pi_2(r^{-1}n) + \rho(bra) c_k^{-1} \pi_{e_k a} j' \sigma_2 \Pi_3(r^{-1}n) \right) \\ &= \sum_{a \in_k T, r \in L(k)} \rho(bra) \pi_{e_k a} i \Pi_2(r^{-1}n) + \sum_{a \in T_k, r \in L(k)} \rho(bra) \pi_{e_k a} j' \sigma_2 \Pi_3(r^{-1}n) \\ &= b \sum_{a \in_k T, r \in L(k)} r a \pi_{e_k a} (r^{-1} \Pi_2(n)) + b \sum_{a \in_k T, r \in L(k)} r a \pi_{e_k a} (r^{-1} \sigma_2 \Pi_3(n)) \\ &= b \sum_{a \in_k T, r \in L(k)} r a \pi_{ra} (\Pi_2 n) + b \sum_{a \in_k T, r \in L(k)} r a \pi_{ra} (\sigma_2 \Pi_3(n)) \\ &= b \sum_{a \in_k T, r \in L(k)} \alpha \xi_{ra} \pi_{ra} (\Pi_2 n) + b \sum_{a \in_k T, r \in L(k)} \alpha \xi_{ra} \pi_{ra} (\sigma_2 \Pi_3(n)) \\ &= b \alpha (\Pi_2 n) + b \alpha (\sigma_2 \Pi_3(n)) \\ &= 0 \end{aligned}$$

by Lemma 1. This completes the proof that \overline{N} is annihilated by $R(\tilde{P})$. \square

Definition 11. We will refer to $\widetilde{\mu}_k(\mathcal{N}) = (\overline{N}, \overline{V})$ as the premutated decorated representation.

As in [6, Proposition 10.9] we now show that the isoclass of the premutated decorated representation does not depend on the choice of the splitting data.

Proposition 4. *The isoclass of the decorated representation $\widetilde{\mu}_k(\mathcal{N}) = (\overline{N}, \overline{V})$ does not depend on the choice of the splitting data.*

Proof. Suppose that we fix $p : N_{out} \rightarrow \ker(\gamma)$ such that $pj = id_{\ker(\gamma)}$ where $j : \ker(\gamma) \rightarrow N_{out}$ is the inclusion map. Let $p' : N_{out} \rightarrow \ker(\gamma)$ be another map satisfying $p'j = id_{\ker(\gamma)}$. Then the restriction of the map $p' - p$ to the subspace $\ker(\gamma)$ is the zero map. Since $\gamma : N_{out} \rightarrow N_{in}$ then $N_{out}/\ker(\gamma) \cong im(\gamma)$.

Consider the following sequence of maps:

$$\ker(\gamma) \xrightarrow{j} N_{out} \xrightarrow{\gamma} N_{out}/\ker(\gamma) \cong im(\gamma)$$

By the universal property of the cokernel of j , there exists a unique linear map $\xi : im(\gamma) \rightarrow ker(\gamma)$ making the following diagram commute

$$\begin{array}{ccccc} ker(\gamma) & \xrightarrow{j} & N_{out} & \xrightarrow{\gamma'} & im(\gamma) \\ & & \downarrow p'-p & \nearrow \xi & \\ & & ker(\gamma) & & \end{array}$$

It follows that $p' = p + \xi\gamma'$ for some linear map $\xi : im(\gamma) \rightarrow ker(\gamma)$.

Now suppose that we fix a map $\sigma_2 : ker(\alpha)/im(\gamma) \rightarrow ker(\alpha)$ such that $\pi_2\sigma_2 = id_{ker(\alpha)/im(\gamma)}$. Let $\sigma'_2 : ker(\alpha)/im(\gamma) \rightarrow ker(\alpha)$ be another map satisfying $\pi_2\sigma'_2 = id_{ker(\alpha)/im(\gamma)}$. By the universal property of the kernel of π_2 , there exists a unique linear map $\eta : ker(\alpha)/im(\gamma) \rightarrow im(\gamma)$ making the following diagram commute

$$\begin{array}{ccccc} ker(\alpha)/im(\gamma) & & & & \\ \downarrow \eta & \searrow \sigma'_2 - \sigma_2 & & & \\ im(\gamma) & \xrightarrow{\quad} & ker(\alpha) & \xrightarrow{\pi_2} & ker(\alpha)/im(\gamma) \end{array}$$

Thus $\sigma'_2 = \sigma_2 + \eta$ for some linear map $\eta : ker(\alpha)/im(\gamma) \rightarrow im(\gamma)$.

Let $\overline{N'}(a^*)$ be the map in 4.3 with σ_2 replaced by σ'_2 . Similarly, let $\overline{N'}(*b)$ be the map in 4.4 with p replaced by p' .

As in [6, Proposition 10.9] we now construct a linear automorphism $\psi : \overline{N}_k \rightarrow \overline{N'}_k$ such that $\overline{N}(a^*) = \overline{N'}(a^*)\psi$ and $\psi\overline{N}(*b) = \overline{N'}(*b)$. Since $\overline{N}_k = \frac{ker(\gamma)}{im(\beta)} \oplus im(\gamma) \oplus \frac{ker(\alpha)}{im(\gamma)} \oplus V_k$, then we may realize ψ as a matrix of order 4. Define ψ as

$$\psi = \begin{pmatrix} I & \pi_1\xi & 0 & 0 \\ 0 & I & -\eta & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

where I is the identity transformation. Note that ψ is invertible. We have

$$\begin{aligned} \overline{N'}(a^*)\psi &= \begin{pmatrix} 0 & c_k^{-1}\pi_{e_ka}i & c_k^{-1}\pi_{e_ka}j'\sigma'_2 & 0 \end{pmatrix} \begin{pmatrix} I & \pi_1\xi & 0 & 0 \\ 0 & I & -\eta & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \\ &= \begin{pmatrix} 0 & c_k^{-1}\pi_{e_ka}i & -c_k^{-1}\pi_{e_ka}i\eta + c_k^{-1}\pi_{e_ka}j'\sigma'_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & c_k^{-1}\pi_{e_ka}i & c_k^{-1}\pi_{e_ka}(-i\eta + j'\sigma'_2) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & c_k^{-1}\pi_{e_ka}i & c_k^{-1}\pi_{e_ka}(-i\eta + j'\sigma_2 + j'\eta) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & c_k^{-1}\pi_{e_ka}i & c_k^{-1}\pi_{e_ka}j'\sigma_2 & 0 \end{pmatrix} \\ &= \overline{N}(a^*) \end{aligned}$$

On the other hand:

$$\begin{aligned}
 \psi \overline{N}(*b) &= \begin{pmatrix} -\pi_1(p + \xi\gamma')\xi_{be_k} \\ -\gamma'\xi_{be_k} \\ 0 \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} -\pi_1 p' \xi_{be_k} \\ -\gamma' \xi_{be_k} \\ 0 \\ 0 \end{pmatrix} \\
 &= \overline{N'}(*b)
 \end{aligned}$$

Now let $\varphi : \overline{N} \rightarrow \overline{N'}$ be the map defined as $\varphi_j = id$ if $j \neq k$ and $\varphi_k = \psi$. Suppose first that $a \in_k T$, $d_1 \in D_{\tau(a)}$, $d_2 \in D_k$ and $w \in \overline{N}_k$. Then

$$\begin{aligned}
 \varphi(d_1 a^* d_2 w) &= d_1 a^* d_2 w \\
 &= d_1 \overline{N}(a^*)(d_2 w) \\
 &= d_1 \overline{N'}(a^*)\psi(d_2 w) \\
 &= d_1 a^* d_2 \varphi(w)
 \end{aligned}$$

Now if $b \in T_k$, $d_1 \in D_k$, $d_2 \in D_{\sigma(b)}$ and $n \in N_{\sigma(b)}$. Then

$$\begin{aligned}
 \varphi(d_1(*b)d_2 n) &= \psi(d_1(*b)d_2 n) \\
 &= \psi(d_1 \overline{N}(*b)(d_2 n)) \\
 &= d_1 \psi(\overline{N}(*b)(d_2 n)) \\
 &= d_1 \overline{N'}(*b)(d_2 n) \\
 &= d_1(*b)d_2 \varphi(n)
 \end{aligned}$$

Therefore for each $u \in \mathcal{F}_S(\mu_k M)$ we obtain a commutative diagram:

$$\begin{array}{ccc}
 \overline{N} & \xrightarrow{u_{\overline{N}}} & \overline{N} \\
 \downarrow \varphi & & \downarrow \varphi \\
 \overline{N'} & \xrightarrow{u_{\overline{N'}}} & \overline{N'}
 \end{array}$$

This proves that the decorated representations $\widetilde{\mu}_k(\mathcal{N}) = (\overline{N}, \overline{V})$ and $(\overline{N'}, \overline{V})$ are right-equivalent, as desired. \square

Let $\mathcal{N} = (N, V)$ be a decorated representation of $(\mathcal{F}_S(M), P)$ and let $\mathcal{N}' = (N', V')$ be a decorated representation of $(\mathcal{F}_S(M'), P')$. Suppose that such representations are right-equivalent, then there exists an algebra isomorphism $\varphi : \mathcal{F}_S(M) \rightarrow \mathcal{F}_S(M')$ such that $\varphi(P)$ is cyclically equivalent to P' . By [2, Theorem 5.3] we have: $R(P') = R(\varphi(P)) = \varphi(R(P))$. Using the representation $\mathcal{N} = (N, V)$ we construct a decorated representation $\widehat{\mathcal{N}} = (\widehat{N}, V)$ of $(\mathcal{F}_S(M'), \varphi(P))$ as follows: let $\widehat{N} = N$ as F -vector spaces and given $u \in \mathcal{F}_S(M')$ and $n \in N$ define $u * n := \varphi^{-1}(u)n$. Clearly $R(P')\widehat{N} = 0$. We will denote \widehat{N} by $\widehat{N} = \varphi^{-1} N$.

Proposition 5. *Let $\varphi : \mathcal{F}_S(M) \rightarrow \mathcal{F}_S(M)$ be a unitriangular automorphism and let $\mathcal{N} = (N, V)$ be a decorated representation of $(\mathcal{F}_S(M), P)$ where P is a potential in $\mathcal{F}_S(M)$ such that $e_k P e_k = 0$. Then:*

- (a) *There exists a unitriangular automorphism $\hat{\varphi} : \mathcal{F}_S(\mu_k M) \rightarrow \mathcal{F}_S(\mu_k M)$ such that $\hat{\varphi}(\mu_k P)$ is cyclically equivalent to $\mu_k(\varphi(P))$.*
- (b) *There exists an isomorphism of decorated representations $\psi : \widetilde{\mu_k}(\mathcal{N}) \rightarrow \widetilde{\mu_k}(\widehat{\mathcal{N}})$.*

Proof. The fact that (a) holds is an immediate consequence of [2, Theorem 8.12]. Let us show (b). Let $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ be the maps associated to the representation $\widehat{\mathcal{N}}$. Recall that ${}_k \hat{T} = \{sa : a \in {}_k T, s \in L(k)\}$ is a local basis for $(e_k M)_S$. We have that $\varphi(sa) = \sum_{r \in L(k), a_1 \in {}_k T} r a_1 C_{ra_1, sa}$ for some $C_{ra_1, sa} \in e_{\tau(a_1)} \mathcal{F}_S(M) e_{\tau(a)}$.

Define $C : N_{in} \rightarrow N_{in}$ as the F -linear map such that for all $r, s \in L(k)$, $a, a_1 \in {}_k T$, the map:

$$\pi_{ra_1} C_{\xi_{sa}} : N_{\tau(a)} \rightarrow N_{\tau(a_1)}$$

is given by

$$\pi_{ra_1} C_{\xi_{sa}}(n) = \varphi^{-1}(C_{ra_1, sa})n$$

for every $n \in N_{\tau(a)}$. Let us show that $\hat{\alpha}C = \alpha$. It suffices to show that for all $a \in {}_k T, r \in L(k)$ we have $\hat{\alpha}C\xi_{ra} = \alpha\xi_{ra}$.

In what follows, given $h \in \mathcal{F}_S(M)$ and $n \in N$ then $h * n = \varphi^{-1}(h)n$ denotes the product in \widehat{N} .

We have

$$\begin{aligned} \hat{\alpha}C\xi_{ra}(n) &= \sum_{s \in L(k), a_1 \in {}_k T} \hat{\alpha}\xi_{sa_1} \pi_{sa_1} C_{\xi_{ra}}(n) \\ &= \sum_{s \in L(k), a_1 \in {}_k T} \hat{\alpha}\xi_{sa_1} (\varphi^{-1}(C_{sa_1, ra})n) \\ &= \sum_{s \in L(k), a_1 \in {}_k T} sa_1 * \varphi^{-1}(C_{sa_1, ra})n \\ &= \varphi^{-1} \left(\sum_{s \in L(k), a_1 \in {}_k T} sa_1 C_{sa_1, ra} \right) n \\ &= \varphi^{-1}(\varphi(ra))n \\ &= ran \\ &= \alpha\xi_{ra}(n) \end{aligned}$$

and therefore $\hat{\alpha}C = \alpha$. This yields the following equalities:

$$(4.5) \quad \begin{aligned} \ker(\alpha) &= C^{-1}(\ker(\hat{\alpha})) \\ \text{im}(\alpha) &= \text{im}(\hat{\alpha}) \end{aligned}$$

Similarly, for each $b \in T_k$ and $s \in L(k)$:

$$\varphi(bs) = \sum_{r \in L(k), b_1 \in T_k} D_{bs, b_1 r} b_1 r$$

for some $D_{bs,b_1r} \in e_{\sigma(b)}\mathcal{F}_S(M)e_{\sigma(b_1)}$.

Thus there exists an F -linear map $D : N_{out} \rightarrow N_{out}$ such that for all $r, s \in L(k)$, $b, b_1 \in T_k$ we have:

$$\pi_{bs}D\xi_{b_1r}(n) = \varphi^{-1}(D_{bs,b_1r})n$$

for every $n \in N_{\sigma(b_1)}$. We now show that $D\hat{\beta} = \beta$. It suffices to show that for all $b \in T_k$, $s \in L(k)$ we have $\pi_{bs}D\hat{\beta} = \pi_{bs}\beta$. Let $n \in N_k$, then

$$\begin{aligned} \pi_{bs}D\hat{\beta}(n) &= \sum_{r \in L(k), b_1 \in T_k} \pi_{bs}D\xi_{b_1r}\pi_{b_1r}\hat{\beta}(n) \\ &= \sum_{r \in L(k), b_1 \in T_k} \varphi^{-1}(D_{bs,b_1r})((b_1r) * n) \\ &= \sum_{r \in L(k), b_1 \in T_k} \varphi^{-1}(D_{bs,b_1r})\varphi^{-1}(b_1r)n \\ &= \varphi^{-1}\left(\sum_{r \in L(k), b_1 \in T_k} D_{bs,b_1r}b_1r\right)n \\ &= \varphi^{-1}(\varphi(bs))n \\ &= bsn \\ &= \pi_{bs}\beta(n) \end{aligned}$$

Therefore $D\hat{\beta} = \beta$, as claimed. Then we obtain the following equalities

$$\begin{aligned} \text{im}(\beta) &= D(\text{im}(\hat{\beta})) \\ \text{ker}(\beta) &= \text{ker}(\hat{\beta}) \end{aligned} \tag{4.6}$$

Lemma 4. *We have that $\hat{\gamma} = C\gamma D$.*

Proof. Using [2, Lemma 9.2] we obtain an algebra isomorphism:

$$\rho : \mathcal{F}_S(M)_{\hat{k}, \hat{k}} \rightarrow \mathcal{F}_S((\mu_k M)_{\hat{k}, \hat{k}})$$

we may view ρ as a monomorphism of algebras:

$$\rho : \bar{e}_k \mathcal{F}_S(M) \bar{e}_k \rightarrow \bar{e}_k \mathcal{F}_S(\mu_k M) \bar{e}_k$$

By [2, Proposition 8.11] we have algebra isomorphisms:

$$\begin{aligned} \phi : \mathcal{F}_S(\widehat{M}) &\rightarrow \mathcal{F}_S(\widehat{M}) \\ \hat{\varphi} : \mathcal{F}_S(\mu_k M) &\rightarrow \mathcal{F}_S(\mu_k M) \end{aligned}$$

where $\widehat{M} := M \oplus (e_k M)^* \oplus (M e_k)$ and

$$\begin{aligned} i_M : \mathcal{F}_S(M) &\rightarrow \mathcal{F}_S(\widehat{M}) \\ i_{\mu_k M} : \mathcal{F}_S(\mu_k M) &\rightarrow \mathcal{F}_S(\widehat{M}) \end{aligned}$$

are the inclusion maps. We also have commutative diagrams

$$\begin{array}{ccc}
 \mathcal{F}_S(\mu_k M) & \xrightarrow{\hat{\varphi}} & \mathcal{F}_S(\mu_k M) \\
 \downarrow i_{\mu_k M} & & \downarrow i_{\mu_k M} \\
 \mathcal{F}_S(\widehat{M}) & \xrightarrow{\phi} & \mathcal{F}_S(\widehat{M})
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{F}_S(M) & \xrightarrow{\varphi} & \mathcal{F}_S(M) \\
 \downarrow i_M & & \downarrow i_M \\
 \mathcal{F}_S(\widehat{M}) & \xrightarrow{\phi} & \mathcal{F}_S(\widehat{M})
 \end{array}$$

Let us see that the previous diagram induces a commutative diagram:

$$\begin{array}{ccc}
 \mathcal{F}_S(M)_{\hat{k}, \hat{k}} & \xrightarrow{\rho} & \mathcal{F}_S((\mu_k M)_{\hat{k}, \hat{k}}) \\
 \downarrow \varphi & & \downarrow \hat{\varphi} \\
 \mathcal{F}_S(M)_{\hat{k}, \hat{k}} & \xrightarrow{\rho} & \mathcal{F}_S((\mu_k M)_{\hat{k}, \hat{k}})
 \end{array}$$

Indeed, on one hand $i_{\mu_k M} \rho \varphi = i_M \varphi$ and on the other hand:

$$i_{\mu_k M} \hat{\varphi} \rho = \phi i_{\mu_k M} \rho = \phi i_M = i_M \varphi$$

Since $i_{\mu_k M}$ is injective then $\rho \varphi = \hat{\varphi} \rho$.

Let $\hat{\Delta} : T_S(\mu_k M) \rightarrow T_S(\mu_k M) \otimes_Z T_S(\mu_k M)$ be the derivation associated to $T_S(\mu_k M)$. Define maps:

$$\begin{aligned}
 \rho^k &: \bar{e}_k T_S(M) \rightarrow T_S(\mu_k M) \\
 {}^k \rho &: T_S(M) \bar{e}_k \rightarrow T_S(\mu_k M)
 \end{aligned}$$

as follows, $\rho^k(z) = \rho(z \bar{e}_k)$ and ${}^k \rho(z) = \rho(\bar{e}_k z)$.

Lemma 5. *For $z \in T_S(M)_{\hat{k}, \hat{k}}$ we have that $\hat{\Delta} \rho(z) = (\rho^k \otimes ({}^k \rho)) \Delta(z)$.*

Proof. The $T_S(\mu_k M)$ -bimodule $T_S(\mu_k M) \otimes_Z T_S(\mu_k M)$ is a $T_S(M)_{\hat{k}, \hat{k}}$ -bimodule via the map ρ . We have that $\hat{\Delta}$ is a $T_S(M)_{\hat{k}, \hat{k}}$ -derivation, $\rho^k \otimes ({}^k \rho)$ is a map of $T_S(M)_{\hat{k}, \hat{k}}$ -bimodules and Δ is a derivation of $T_S(M)$. Therefore $\hat{\Delta} \rho$ and $(\rho^k \otimes ({}^k \rho)) \Delta$ are derivations of $T_S(\mu_k M)$. Since $T_S(M)_{\hat{k}, \hat{k}}$ is generated, as an F -algebra, by $\bar{e}_k S$, $\bar{e}_k M_0 \bar{e}_k$ and $M_0 D_k M_0$, then it suffices to establish the equality for $z \in \bar{e}_k S \cup \bar{e}_k M_0 \bar{e}_k \cup M_0 D_k M_0$.

If $z \in \bar{e}_k S$, then

$$\hat{\Delta} \rho(z) = 1 \otimes \rho(z) - \rho(z) \otimes 1 = \bar{e}_k \otimes \rho(z) - \rho(z) \otimes \bar{e}_k = (\rho^k \otimes ({}^k \rho)) \Delta(z)$$

For $z \in \bar{e}_k M_0 \bar{e}_k$ we have

$$\hat{\Delta} \rho(z) = 1 \otimes \rho(z) = \bar{e}_k \otimes \rho(z) = (\rho^k \otimes ({}^k \rho)) \Delta(z)$$

If $z = m_1 r m_2$ where $m_1 \in \bar{e}_k M_0 e_k$, $r \in D_k$ and $m_2 \in e_k M_0 \bar{e}_k$, then

$$\hat{\Delta} \rho(m_1 r m_2) = 1 \otimes \rho(m_1 r m_2) = \bar{e}_k \otimes \rho(m_1 r m_2)$$

and

$$\begin{aligned}
(\rho^k \otimes {}^k \rho) \Delta(m_1 r m_2) &= (\rho^k \otimes ({}^k \rho))(\Delta(m_1) r m_2 + m_1 \Delta(r m_2)) \\
&= (\rho^k \otimes {}^k \rho)(1 \otimes m_1 r m_2 + m_1 \otimes r m_2) \\
&= \bar{e}_k \otimes \rho(m_1 r m_2) + m_1 \bar{e}_k \otimes ({}^k \rho(r m_2)) \\
&= \bar{e}_k \otimes \rho(m_1 r m_2)
\end{aligned}$$

completing the proof of Lemma 5. \square

Lemma 6. For $\alpha \in T_S(M)_{\hat{k}, \hat{k}}$, $z \in \mathcal{F}_S(M)_{\hat{k}, \hat{k}}$ we have

$$((\rho^k \otimes ({}^k \rho)) \Delta(\alpha)) \diamond \rho(z) = \rho^k(\Delta(\alpha) \diamond z)$$

Proof. One can verify that if the equality holds for all α and for all z , and for all β and every z then it holds for all $\alpha\beta$ and every z . Therefore it suffices to establish the equality for $\alpha \in \bar{e}_k S \cup \bar{e}_k M_0 \bar{e}_k \cup M_0 D_k M_0$.

(i) Suppose first that $\alpha \in \bar{e}_k S$, then

$$\begin{aligned}
(\rho^k \otimes ({}^k \rho)) \Delta(\alpha) \diamond \rho(z) &= (\rho^k \otimes ({}^k \rho))(1 \otimes \alpha - \alpha \otimes 1) \diamond \rho(z) \\
&= (\bar{e}_k \otimes \rho(\alpha) - \rho(\alpha) \otimes \bar{e}_k) \diamond \rho(z) \\
&= cyc(\rho(\alpha) \rho(z) - \rho(z) \rho(\alpha)) \\
&= \rho(cyc(\alpha z - z \alpha)) \\
&= \rho^k(\Delta(\alpha) \diamond z)
\end{aligned}$$

(ii) If $\alpha \in \bar{e}_k M_0 \bar{e}_k$, then

$$\begin{aligned}
(\rho^k \otimes ({}^k \rho)) \Delta(\alpha) \diamond \rho(z) &= (\rho^k \otimes ({}^k \rho))(1 \otimes \alpha) \diamond \rho(z) \\
&= (\bar{e}_k \otimes p(\alpha)) \diamond \rho(z) \\
&= cyc(\rho(\alpha) \rho(z)) \\
&= \rho(cyc(\alpha z)) \\
&= \rho^k(\Delta(\alpha) \diamond z)
\end{aligned}$$

(iii) Finally, if $\alpha = m_1 r m_2$ where $m_1 \in \bar{e}_k M_0 e_k$, $r \in D_k$ and $m_2 \in e_k M_0 \bar{e}_k$, then

$$\begin{aligned}
(\rho^k \otimes ({}^k \rho)) \Delta(m_1 r m_2) \diamond \rho(z) &= (\rho^k \otimes ({}^k \rho))(1 \otimes m_1 r m_2 + m_1 \otimes r m_2) \diamond \rho(z) \\
&= (\bar{e}_k \otimes \rho(m_1 r m_2)) \diamond \rho(z) \\
&= cyc(\rho(m_1 r m_2 z)) \\
&= \rho^k(\Delta(m_1 r m_2) \diamond z)
\end{aligned}$$

\square

Lemma 5 and Lemma 6 imply immediately the following

Lemma 7. Let $\alpha \in T_S(M)_{\hat{k}, \hat{k}}$, $z \in \mathcal{F}_S(M)_{\hat{k}, \hat{k}}$, then:

$$\hat{\Delta} \rho(\alpha) \diamond \rho(z) = \rho^k(\Delta(\alpha) \diamond z)$$

Let $r, s \in L(k)$, $a \in {}_k T$ and $b \in T_k$. Let $n \in \hat{N}_{\sigma(b)}$, then

$$\hat{\gamma}_{sa, br}(n) = \sum_{w \in L(k)} s^*(r^{-1} w) \varphi^{-1} \left(Y_{[bwa]}(\varphi(P)) \right) n.$$

We have

$$\begin{aligned}\varphi^{-1} \left(Y_{[bwa]}(\varphi(P)) \right) n &= \varphi^{-1} \rho^{-1} \left(X_{[bwa]^*}(\rho\varphi(P)) \right) n \\ &= \left(\rho^{-1} \hat{\varphi}^{-1} X_{[bwa]^*}(\hat{\varphi}(\rho(P))) \right) n\end{aligned}$$

Also $X_{[bwa]^*}(\hat{\varphi}(\rho(P))) = \lim_{u \rightarrow \infty} Z_u$ where:

$$Z_u = \sum_{s \in L(\sigma(b))} \sum_{c \in \tilde{T}_{\hat{k}, \hat{k}}} (s\rho(bwa))^* \left(\hat{\Delta}(\hat{\varphi}(c)^{\leq u+1}) \diamond \hat{\varphi}(X_{c^*}(\rho(P))) \right) s$$

Let v be an arbitrary positive integer and let $\alpha, \beta \in \mathcal{F}_S(M)$. We will write $\alpha \equiv \beta (v)$ if $\alpha - \beta \in \mathcal{F}_S(M)^{>v}$.

Clearly for any $\alpha \in \mathcal{F}_S(M)$ and for every positive integer v we have $\alpha \equiv \alpha^{\leq v} (v)$.

Note that if $h \in T_S(M)^{>v}$ and $z \in \mathcal{F}_S(M)$, then $\Delta(h) \diamond z \in \mathcal{F}_S(M)^{>v}$. Therefore if $\alpha, \beta \in T_S(M)$ and $\alpha \equiv \beta (v)$, then $\Delta(\alpha) \diamond z \equiv \Delta(\beta) \diamond z (v)$ for every $z \in \mathcal{F}_S(M)$.

Let $h \in \mathcal{F}_S(M)_{\hat{k}, \hat{k}}$. Let us see that:

$$\rho(h^{\leq 2v+3}) \equiv \rho(h)^{\leq v+1} (v+2)$$

Indeed, since $h - h^{\leq 2v+3} \in \mathcal{F}_S(M)^{\geq 2v+4}$ then $\rho(h) - \rho(h^{\leq 2v+3}) \in \mathcal{F}_S(\mu_k M)^{\geq v+2}$, whence $\rho(h) \equiv \rho(h^{\leq 2v+3}) (v+2)$ and therefore $\rho(h) \equiv \rho(h)^{\leq v+1} (v+2)$.

Let $v \gg 0$ be such that $\mathcal{F}_S(M)^{\geq v} N = 0$.

For $i \geq 1$ we have that $Z_{v+i} - Z_v \in \mathcal{F}_S(\mu_k M)^{\geq v+1}$ and thus

$$\varphi^{-1} \rho^{-1}(Z_{v+i}) \equiv \varphi^{-1} \rho^{-1}(Z_v) (v+1)$$

therefore

$$\lim_{u \rightarrow \infty} \varphi^{-1} \rho^{-1}(Z_u) n = \varphi^{-1} \rho^{-1}(Z_v) n$$

Define:

$$W(c) = \varphi^{-1} \rho^{-1} \left(\sum_{s \in L(\sigma(b))} (s\rho(bwa))^* \left(\hat{\Delta}(\hat{\varphi}(c)^{\leq v+1}) \diamond \hat{\varphi}(X_{c^*}(\rho(P))) \right) s \right) n.$$

We require the following:

Lemma 8. *If $c \in \bar{e}_k M_0 \bar{e}_k \cap T$ then $W(c) = 0$.*

Proof. Note that $X_{c^*}(\rho(P)) = \rho(X_{c^*}(P))$, hence

$$\hat{\varphi}(c)^{\leq v+1} = \hat{\varphi}(\rho(c))^{\leq v+1} = \rho(\varphi(c))^{\leq v+1} \equiv \rho(\varphi(c))^{\leq 2v+3} (v+2)$$

Consequently

$$\begin{aligned} W(c) &= \varphi^{-1} \rho^{-1} \left(\sum_{s \in L(\sigma(b))} (s\rho(bwa))^* \left(\hat{\Delta}(\rho(\varphi(c)^{\leq 2v+3})) \diamond \hat{\varphi} \rho(X_{c^*}(P)) \right) s \right) n \\ &= \varphi^{-1} \rho^{-1} \left(\sum_{s \in L(\sigma(b))} (s\rho(bwa))^* \left(\hat{\Delta}(\rho(\varphi(c)^{\leq 2v+3})) \diamond \rho \varphi(X_{c^*}(P)) \right) s \right) n \end{aligned}$$

Using Lemma 7 we obtain

$$W(c) = \varphi^{-1} \rho^{-1} \left(\sum_{s \in L(\sigma(b))} (s\rho(bwa))^* \left(\rho^k \left(\Delta(\varphi(c)^{\leq 2v+3}) \diamond \varphi(X_{c^*}(P)) \right) \right) s \right) n$$

Letting $z = \varphi(X_{c^*}(P))$ yields that $W(c)$ is a sum of elements of the form:

$$\varphi^{-1} \rho^{-1} \left(\sum_{s \in L(\sigma(b))} (s\rho(bwa))^* \left(\rho^k (\Delta(m_1 \dots m_l r) \diamond z) \right) s \right) n$$

where $m_1, \dots, m_l \in SM_0$ and $r \in \bar{e}_k S$.

Then

$$\sum_{s \in L(\sigma(b))} (s\rho(bwa))^* \left(\rho^k (\Delta(m_1 \dots m_l r) \diamond z) \right) s$$

is equal to

$$\sum_{s \in L(\sigma(b))} (s\rho(bwa))^* \left(\rho^k (\Delta(m_1 \dots m_l) r \diamond z) \right) s + g$$

where

$$\begin{aligned} g &= \sum_{s \in L(\sigma(b))} (s\rho(bwa))^* \left(\rho^k (m_1 \dots m_l \Delta(r) \diamond z) \right) s \\ &= \sum_{s \in L(\sigma(b))} (s\rho(bwa))^* (r\rho(\text{cyc}(\bar{e}_k z m_1 \dots m_l)) - \rho(\text{cyc}(\bar{e}_k z m_1 \dots m_l))r) s \end{aligned}$$

By [2, Lemma 5.2] we obtain that $g = 0$. Therefore:

$$W(c) = \varphi^{-1} \rho^{-1} \left(\sum_{s \in L(\sigma(b))} (s\rho(bwa))^* \left(\rho^k (\Delta(m_1 \dots m_l) \diamond r z) \right) s \right) n$$

then $W(c)$ is a sum of elements of the form:

$$\varphi^{-1} \rho^{-1} ((s\rho(bwa))^* (\rho(\bar{e}_k m_i \dots m_l r z m_1 \dots m_{i-1})) s) n$$

If $i = l$ then $z = \sum_i z_i z_{i'}$ where $z_i \in \bar{e}_k M$ and in this case:

$$(s\rho(bwa))^* (\rho(\bar{e}_k m_l r \bar{e}_k z m_1 \dots m_{l-1})) = (s\rho(bwa))^* (\rho(\bar{e}_k m_l r) \rho(\bar{e}_k z m_1 \dots m_{l-1} \bar{e}_k)) = 0$$

Hence $W(c)$ is a sum of elements of the form:

$$\varphi^{-1} \rho^{-1} ((s\rho(bwa))^* (\rho(\bar{e}_k m_i m_{i+1}) \rho(\alpha z \beta)) s) n$$

where $m_i m_{i+1} \in Me_k M$ and thus $W(c)$ is a sum of elements of the form $\varphi^{-1}(\alpha)\varphi^{-1}(z)\varphi^{-1}(\beta)n$. Since $\varphi^{-1}(z) = X_{c^*}(P)$ then $W(c) \in R(P)N = 0$, completing the proof of Lemma 8. \square

From the above we obtain the following formula:

$$(*) : \varphi^{-1}(Y_{[bwa]}(\varphi(P)))n = \varphi^{-1}\rho^{-1}(Z'_v)n$$

where:

$$\begin{aligned} Z'_v &= \sum_{s \in L(\sigma(b))} \sum_{b_1 \in T_k, r \in L(k), a_1 \in_k T} (s\rho(bwa))^* \left(\hat{\Delta}(\hat{\varphi}(\rho(b_1 r a_1)))^{\leq v+1} \diamond \hat{\varphi}(X_{\rho(b_1 r a_1)}(\rho(P))) \right) s \\ &= \sum_{s \in L(\sigma(b))} \sum_{b_1 r a_1} (s\rho(bwa))^* \left(\hat{\Delta}(\hat{\varphi}(\rho(b_1 r a_1)))^{\leq v+1} \diamond \hat{\varphi}\rho(Y_{[b_1 r a_1]}(P)) \right) s \\ &= \sum_{s \in L(\sigma(b))} \sum_{b_1 r a_1} (s\rho(bwa))^* \left(\hat{\Delta}(\rho\varphi(b_1 r a_1))^{\leq v+1} \diamond \rho\varphi(Y_{[b_1 r a_1]}(P)) \right) s \end{aligned}$$

Thus, in $(*)$ we may replace Z'_v by the term:

$$\sum_{b_1 r a_1} \sum_{s \in L(\sigma(b))} (s\rho(bwa))^* \left(\hat{\Delta}(\rho(\varphi(b_1 r a_1))^{\leq 2v+3}) \diamond \rho\varphi(Y_{[b_1 r a_1]}(P)) \right) s$$

which by Lemma 7 is equal to:

$$\sum_{b_1 r a_1} \sum_{s \in L(\sigma(b))} (s\rho(bwa))^* \left(\rho^k \left(\Delta(\varphi(b_1 r a_1))^{\leq 2v+3} \diamond \varphi(Y_{[b_1 r a_1]}(P)) \right) \right) s$$

the latter term can be replaced by:

$$\sum_{b_1 r a_1} \sum_{s \in L(\sigma(b))} (s\rho(bwa))^* \left(\rho^k \left(\Delta(\varphi(b_1 r))^{\leq 2v+3} \varphi(a_1)^{\leq 2v+3} \diamond \varphi(Y_{[b_1 r a_1]}(P)) \right) \right) s$$

which in turn can be replaced by $S = S_1 + S_2$, where:

$$\begin{aligned} S_1 &= \sum_{s \in L(\sigma(b))} \sum_{b_1 r a_1} (s\rho(bwa))^* \left(\rho^k \left(\Delta(\varphi(b_1))^{\leq 2v+3} \diamond \varphi(r a_1 Y_{[b_1 r a_1]}(P)) \right) \right) s \\ S_2 &= \sum_{s \in L(\sigma(b))} \sum_{b_1 r a_1} (s\rho(bwa))^* \left(\rho^k \left(\Delta(\varphi(a_1))^{\leq 2v+3} \diamond \varphi(Y_{[b_1 r a_1]}(P) b_1 r) \right) \right) s \end{aligned}$$

Using 3.7 gives that:

$$S_2 = \sum_{s \in L(\sigma(b))} \sum_{a_1} (s\rho(bwa))^* \left(\rho^k \left(\Delta(\varphi(a_1))^{\leq 2v+3} \diamond \varphi(X_{a_1^*}(P)) \right) \right) s$$

whence:

$$\varphi^{-1}(Y_{[bwa]}(\varphi(P)))n = \varphi^{-1}\rho^{-1}(S_1)n + \varphi^{-1}\rho^{-1}(S_2)n$$

(i) Let us see that $\varphi^{-1}\rho^{-1}(S_2) \subseteq R(P)$.

Let $z = \varphi(X_{a_1^*}(P))$, then $\varphi(a_1)^{\leq 2v+3}$ is a sum of elements of the form $m_1 \dots m_l t$ where $m_j \in SM_0$, $t \in \bar{e}_k S$. Therefore S_2 is a sum of elements of the form:

$$(s\rho(bwa))^* (\rho(\bar{e}_k m_i \dots m_l t z m_1 \dots m_{i-1}) \bar{e}_k) s$$

If $i = l$, then $(s\rho(bwa))^* (\rho(\bar{e}_k m_i \dots m_l \dots m_{i-1}) \bar{e}_k) s$ is equal to:

$$(s\rho(bwa))^* (\rho(\bar{e}_k m_l \bar{e}_k) \rho(\bar{e}_k t z m_1 \dots m_{l-1}) \bar{e}_k) s = 0$$

If $i < l$ then:

$$(s\rho(bwa))^*(\rho(\bar{e}_k m_i \dots m_l t z m_1 \dots m_{i-1} \bar{e}_k))s = (s\rho(bwa))^*(\rho(\bar{e}_k m_i \dots m_l))\rho(t z m_1 \dots m_{i-1} \bar{e}_k)s$$

Therefore S_2 is a sum of elements of the form $\rho(\alpha z \beta)$, so $\varphi^{-1}\rho^{-1}(S_2)$ is a sum of elements of the form $\varphi^{-1}(\alpha)\varphi^{-1}(z)\varphi^{-1}(\beta)$. Since $\varphi^{-1}(z) = X_{a_1^*}(P)$ then $\varphi^{-1}\rho^{-1}(S_2) \subseteq R(P)$ which establishes (i).

From the above it follows that $\varphi^{-1}(Y_{[bwa]}(\varphi(P)))n = \varphi^{-1}\rho^{-1}(S_1)n$.

(ii) Let us show that $\varphi^{-1}\rho^{-1}(S_1) = \nu_1 + \nu_2$, where $\nu_1 \in R(P)$ and ν_2 is a sum of elements of the form:

$$\sum_{s \in L(\sigma(b))} \sum_{ra_1} (s\rho(bwa))^* \rho(\bar{e}_k m_i \dots m_l z_{ra_1} m_1 \dots m_{i-1} \bar{e}_k) s$$

where $z_{ra_1} = \varphi(ra_1 Y_{[b_1 ra_1]}(P))$.

Note that $\varphi(b_1)^{\leq 2v+3}$ is a sum of elements of the form $m_1 m_2 \dots m_l$ where $m_1, \dots, m_{l-1} \in SM_0$ and $m_l \in \bar{e}_k M e_k$. The element $\varphi(b_1)$ is a sum of elements of the form $m_1 \dots m_l$ where $m_1, \dots, m_{l-1} \in SM_0$ and $m_l \in \bar{e}_k M e_k$. Then $\varphi^{-1}\rho^{-1}(S_1)$ lies in the F -vector space generated by $\varphi^{-1}\rho^{-1}(T_i)$ where T_i is the F -vector space generated by elements of the form:

$$u_i = \sum_{s \in L(\sigma(b))} \sum_{ra_1} (s\rho(bwa))^* \rho(\bar{e}_k m_i \dots m_l z_{ra_1} m_1 \dots m_{i-1} \bar{e}_k) s$$

Let us show that if $i < l$ then $\varphi^{-1}\rho^{-1}(T_i) \subseteq R(P)$. We have that

$$u_i = \sum_{s \in L(\sigma(b))} (s\rho(bwa))^* (\rho(\bar{e}_k m_i \dots m_{l-1})) \rho(m_l w_{b_1} m_1 \dots m_{i-1} \bar{e}_k) s$$

where $w_{b_1} = \varphi(X_{b_1^*}(P))$.

It follows that u_i is a sum of elements of the form $\rho(\alpha w_{b_1} \beta)$ and thus $\varphi^{-1}\rho^{-1}(u_i)$ is a sum of elements of the form $\varphi^{-1}(\alpha)X_{b_1^*}(P)\varphi^{-1}(\beta)$, as was to be shown. This completes the proof of (ii).

We have that

$$\varphi(b_1) = \sum_{b'r'} D_{b_1, b'r'} b'r'$$

then

$$\varphi(b_1)^{\leq 2v+3} = \sum_{b'r'} (D_{b_1, b'r'})^{\leq 2v+2} b'r'$$

Also

$$\begin{aligned} \varphi(ra_1) &= \sum_{r''a'} r''a' C_{r''a', ra_1} \\ z_{ra_1} &= \sum_{r''a'} r''a' C_{r''a', ra_1} \varphi(Y_{[b_1 ra_1]}(P)) \end{aligned}$$

On the other hand, $(D_{b_1, b'r'})^{\leq 2v+2}$ is a sum of elements of the form $m_1 m_2 \dots m_{l-1} s''$ where each $m_i \in SM_0$ and $s'' \in L(\sigma(b'))$. Therefore $\varphi(b_1)^{\leq 2v+3}$ is a sum of elements of the form $m_1 m_2 \dots m_{l-1} s'' b'r'$.

In what follows, $z(b_1 ra_1) = \varphi(Y_{[b_1 ra_1]}(P))$.

We obtain that $\varphi^{-1}(Y_{[bwa]}(\varphi(P)))n$ is a sum of elements of the form:

$$(*) : \sum_{b', s', s'', r', r''} \varphi^{-1} \rho^{-1} ((s' \rho(bwa))^* (\rho(\bar{e}_k H)))$$

where:

$$\begin{aligned} H &= s'' b' r' r'' a' C_{r'' a', r a_1} z(b_1 r a_1) m_1 \dots m_{l-1} s' n \\ &= \sum_{w_1 \in L(k)} s'' b' w_1^* (r' r'') w_1 a' C_{r'' a', r a_1} z(b_1 r a_1) m_1 \dots m_{l-1} s' n \end{aligned}$$

The non-zero terms of $(*)$ are those with $s' = s''$, $b = b'$, $a' = a$, $w_1 = w$. Thus:

$$\varphi^{-1}(Y_{[bwa]}(\varphi(P)))n = \sum_{h, r', r'', a_1, b_1} w^*(r' r'') \varphi^{-1}(C_{r'' a, h a_1}) Y_{[b_1 h a_1]}(P) \varphi^{-1}(D_{b_1, b r'}) n$$

Therefore:

$$\begin{aligned} \hat{\gamma}_{sa, br}(n) &= \sum_{w \in L(k)} s^*(r^{-1} w) \varphi^{-1}(Y_{[bwa]}(\varphi(P)))n \\ &= \sum_{w, h, r', r'', a_1, b_1} s^*(r^{-1} w) w^*(r' r'') \varphi^{-1}(C_{r'' a, h a_1}) Y_{[b_1 h a_1]}(P) \varphi^{-1}(D_{b_1, b r'}) n \\ &= \sum_{w, h, r', r'', a_1, b_1} s^*(r^{-1} w w^*(r' r'')) \varphi^{-1}(C_{r'' a, h a_1}) Y_{[b_1 h a_1]}(P) \varphi^{-1}(D_{b_1, b r'}) n \\ &= \sum_{h, r', r'', a_1, b_1} s^* \left(r^{-1} \sum_{w \in L(k)} w^*(r' r'') w \right) \varphi^{-1}(C_{r'' a, h a_1}) Y_{[b_1 h a_1]}(P) \varphi^{-1}(D_{b_1, b r'}) n \\ &= \sum_{h, r', r'', a_1, b_1} s^*(r^{-1} r' r'') \varphi^{-1}(C_{r'' a, h a_1}) Y_{[b_1 h a_1]}(P) \varphi^{-1}(D_{b_1, b r'}) n \end{aligned}$$

By [2, Proposition 8.1 (iii)] and [2, Proposition 8.2 (iii)] the following equalities hold:

$$\begin{aligned} \varphi^{-1}(C_{r'' a, h a_1}) &= \sum_{u \in L(\sigma(a))} (r'')^*(hu) \varphi^{-1}(C_{ua, a_1}) \\ \varphi^{-1}(D_{b_1, b r'}) &= \sum_{v \in L(\tau(b_1))} (r')^*(v) \varphi^{-1}(D_{b_1, bv}) \end{aligned}$$

whence

$$\begin{aligned} \hat{\gamma}_{sa, br}(n) &= \sum_{h, r', r'', a_1, b_1, u, v} s^*(r^{-1} r' r'') (r'')^*(hu) \varphi^{-1}(C_{ua, a_1}) Y_{[b_1 h a_1]}(P) (r')^*(v) \varphi^{-1}(D_{b_1, bv}) n \\ &= \sum_{h, r', a_1, b_1, u, v} s^*(r^{-1} r' hu) (r')^*(v) \varphi^{-1}(C_{ua, a_1}) Y_{[b_1 h a_1]}(P) \varphi^{-1}(D_{b_1, bv}) n \\ &= \sum_{h, a_1, b_1, u, v} s^*(r^{-1} v hu) \varphi^{-1}(C_{ua, a_1}) Y_{[b_1 h a_1]}(P) \varphi^{-1}(D_{b_1, bv}) n \end{aligned}$$

On the other hand, using again [2, Proposition 8.1 (iii)] and [2, Proposition 8.2 (iii)] one gets that the (sa, br) -entry of $(C\gamma D)(n)$ is given by

$$\sum_{s', t', h, a_1, b_1, u, v} (s')^*((t')^{-1} h) s^*(s' u) r^*(v t') \varphi^{-1}(C_{ua, a_1}) Y_{[b_1 h a_1]}(P) \varphi^{-1}(D_{b_1, bv}) n$$

Using 2.4 one obtains the following:

$$\begin{aligned}
\sum_{s', t', h, u, v} (s')^* ((t')^{-1} h) s^* (s' u) r^* (v t') &= \sum_{s', t', h, u, v} s^* \left((s')^* ((t')^{-1} h) s' u \right) r^* (v t') \\
&= \sum_{t', h, u, v} s^* \left((t')^{-1} h u \right) r^* (v t') \\
&= \sum_{h, u, v} s^* \left(\sum_{t'} r^* (v t') (t')^{-1} h u \right) \\
&= \sum_{h, u, v} s^* (r^{-1} v h u)
\end{aligned}$$

which implies that $\hat{\gamma} = C\gamma D$ and the proof of Lemma 4 is now complete. \square

Note that Lemma 4 implies the following equalities:

$$\begin{aligned}
(4.7) \quad \ker(\gamma) &= D(\ker(\hat{\gamma})) \\
\operatorname{im}(\gamma) &= C^{-1}(\operatorname{im}(\hat{\gamma}))
\end{aligned}$$

We now complete the proof of Proposition 5. Let us establish a right-equivalence $(\hat{\varphi}, \psi, \eta)$ between the representations $\widetilde{\mu}_k(\mathcal{N})$ and $\widetilde{\mu}_k(\widehat{\mathcal{N}})$. First, we define $\hat{\varphi} : \mathcal{F}_S(\mu_k M) \rightarrow \mathcal{F}_S(\mu_k M)$ as the right-equivalence between the algebras $(\mathcal{F}_S(\mu_k M), \mu_k P)$ and $(\mathcal{F}_S(\mu_k M), \mu_k \varphi(P))$ given by [2, Theorem 8.12]. Let $\widetilde{\mu}_k(\widehat{\mathcal{N}}) = (\widehat{N}, \widehat{V})$. If $i \neq k$, then $\widehat{N}_i = N_i$ and:

$$\widehat{N}_k = \frac{\ker(\hat{\gamma})}{\operatorname{im}(\hat{\beta})} \oplus \operatorname{im}(\hat{\gamma}) \oplus \frac{\ker(\hat{\alpha})}{\operatorname{im}(\hat{\gamma})} \oplus V_k$$

For each $i \in \{1, 2, 3, 4\}$, let \overline{J}_i and $\overline{\Pi}_i$ be the corresponding inclusions and projections associated to \widehat{N}_k , analogous to those given in 4.1 and 4.2. Then we have inclusion maps:

$$\begin{aligned}
\overline{j} : \ker(\hat{\gamma}) &\rightarrow N_{out} \\
\overline{i} : \operatorname{im}(\hat{\gamma}) &\rightarrow N_{in}
\end{aligned}$$

and projections:

$$\begin{aligned}
\overline{\pi}_1 : \ker(\hat{\gamma}) &\rightarrow \frac{\ker(\hat{\gamma})}{\operatorname{im}(\hat{\beta})} \\
\overline{\pi}_2 : \ker(\hat{\alpha}) &\rightarrow \frac{\ker(\hat{\alpha})}{\operatorname{im}(\hat{\gamma})}
\end{aligned}$$

By Lemma 4 we have $\hat{\gamma} = C\gamma D$ and thus $\hat{\gamma} D^{-1} = C\gamma$. It follows that D^{-1} induces an isomorphism

$$D_1^{-1} : \ker(\gamma) \rightarrow \ker(\hat{\gamma})$$

such that $\overline{j} D_1^{-1} = D^{-1} j$. Also, D^{-1} maps $\operatorname{im}(\beta)$ to $\operatorname{im}(\hat{\beta})$. Therefore, D^{-1} also induces an isomorphism

$$\underline{D}^{-1} : \frac{\ker(\gamma)}{\operatorname{im}(\beta)} \rightarrow \frac{\ker(\hat{\gamma})}{\operatorname{im}(\hat{\beta})}$$

such that $\underline{D}^{-1} \pi_1 = \overline{\pi}_1 D_1^{-1}$. The isomorphism C induces an isomorphism

$$C_1 : \text{im}(\gamma) \rightarrow \text{im}(\hat{\gamma})$$

such that $\bar{i}C_1 = Ci$. The equality $\hat{\alpha}C = \alpha$ implies that C also induces an isomorphism $C_2 : \ker(\alpha) \rightarrow \ker(\hat{\alpha})$; thus there exists an isomorphism

$$\underline{C} : \frac{\ker(\alpha)}{\text{im}(\gamma)} \rightarrow \frac{\ker(\hat{\alpha})}{\text{im}(\hat{\gamma})}$$

such that $\underline{C}\pi_2 = \bar{\pi}_2 C_2$.

To construct $\widetilde{\mu}_k(\widehat{N})$ we choose splitting data as follows:

$$\begin{aligned} \bar{p} &= D_1^{-1}pD : N_{out} \rightarrow \ker(\hat{\gamma}) \\ \bar{\sigma}_2 &= C_2\sigma_2\underline{C}^{-1} : \frac{\ker(\hat{\alpha})}{\text{im}(\hat{\gamma})} \rightarrow \ker(\hat{\alpha}) \end{aligned}$$

Note that $\bar{p}\bar{j} = id_{\ker(\hat{\gamma})}$, $\bar{\pi}_2\bar{\sigma}_2 = id_{\ker(\hat{\alpha})/\text{im}(\hat{\gamma})}$. Define:

$$\psi : \bar{N} \rightarrow \widetilde{\bar{N}}$$

as follows. If $i \neq k$ then $\psi_i : \bar{N}_i = N_i \rightarrow \widetilde{\bar{N}}_i = N_i$ is the identity map and

$$\psi_k : \bar{N}_k \rightarrow \widetilde{\bar{N}}_k$$

is the map such that for every $i \neq j$, $\bar{\Pi}_i\psi_k J_j = 0$ and

$$\begin{aligned} \bar{\Pi}_1\psi_k J_1 &= \underline{D}^{-1} \\ \bar{\Pi}_2\psi_k J_2 &= C_1 \\ \bar{\Pi}_3\psi_k J_3 &= \underline{C} \\ \bar{\Pi}_4\psi_k J_4 &= id_{V_k} \end{aligned}$$

Let us show that for every $z \in \tilde{T}$ and $n \in N_{\tau(z)}$:

$$\psi_{\sigma(z)}(zn) = \hat{\varphi}(z)\psi_{\tau(z)}(n)$$

Suppose first that $z = a^*$ where $a \in_k T$. In this case $\tau(z) = \sigma(a) = k$ and $\sigma(z) = \tau(a) \neq k$. By [2, Proposition 8.11]:

$$\hat{\varphi}(a^*) = \sum_{t \in L(k), a_1 \in_k T} (C^{-1})_{a, ta_1} a_1^* t^{-1}$$

whence

$$\widetilde{\bar{N}}(\hat{\varphi}(a^*)) = \sum_{t \in L(k), a_1 \in_k T} (C^{-1})_{a, ta_1} * \widetilde{\bar{N}}(a_1^*) t^{-1}$$

where $*$ denotes the action of $\mathcal{F}_S(M)$ in \widehat{N} . In this case we have to verify the following equality:

$$(4.8) \quad \bar{N}(a^*) = \widetilde{\bar{N}}(\hat{\varphi}(a^*))\psi_k$$

On one hand, $\overline{N}(a^*)J_1 = 0$. On the other hand:

$$\begin{aligned}\overline{\widehat{N}}(\widehat{\varphi}(a^*))\psi_k J_1 &= \sum_{a_1 \in_k T, t \in L(k)} (C^{-1})_{a, ta_1} * \overline{\widehat{N}}(a_1^*)t^{-1}\psi_k J_1 \\ &= \sum_{a_1 \in_k T, t \in L(k)} (C^{-1})_{a, ta_1} * \overline{\widehat{N}}(a_1^*)t^{-1}\overline{J}_1\overline{\Pi}_1\psi_k J_1 \\ &= \sum_{a_1 \in_k T, t \in L(k)} (C^{-1})_{a, ta_1} * \overline{\widehat{N}}(a_1^*)\overline{J}_1 t^{-1}\overline{\Pi}_1\psi_k J_1 = 0\end{aligned}$$

and thus $\overline{N}(a^*)J_1 = \overline{\widehat{N}}(\widehat{\varphi}(a^*))\psi_k J_1$. Now let us consider $\overline{N}(a^*)J_2$. By 4.3 we have $\overline{N}(a^*)J_2 = c_k^{-1}\pi_{e_k a}i$. On the other hand:

$$\begin{aligned}\overline{\widehat{N}}(\widehat{\varphi}(a^*))\psi_k J_2 &= \sum_{a_1 \in_k T, t \in L(k)} (C^{-1})_{a, ta_1} * \overline{\widehat{N}}(a_1^*)t^{-1}\overline{J}_2\overline{\Pi}_2\psi_k J_2 \\ &= \sum_{a_1 \in_k T, t \in L(k)} (C^{-1})_{a, ta_1} * \overline{\widehat{N}}(a_1^*)\overline{J}_2 t^{-1}\overline{\Pi}_2\psi_k J_2 \\ &= \sum_{a_1 \in_k T, t \in L(k)} (C^{-1})_{a, ta_1} * c_k^{-1}\pi_{e_k a_1}\overline{i}t^{-1}\overline{\Pi}_2\psi_k J_2 \\ &= \sum_{a_1 \in_k T, t \in L(k)} (C^{-1})_{a, ta_1} * c_k^{-1}\pi_{e_k a_1}\overline{i}t^{-1}C_1 \\ &= c_k^{-1} \sum_{a_1 \in_k T, t \in L(k)} (C^{-1})_{a, ta_1} * \pi_{e_k a_1}t^{-1}Ci \\ &= c_k^{-1} \sum_{a_1 \in_k T, t \in L(k)} (C^{-1})_{a, ta_1} * \pi_{ta_1}Ci \\ &= c_k^{-1} \sum_{a_1, a_2 \in_k T, r, t \in L(k)} \varphi^{-1}((C^{-1})_{a, ta_1})\pi_{ta_1}C\xi_{ra_2}\pi_{ra_2}i \\ &= c_k^{-1} \sum_{a_1, a_2 \in_k T, r, t \in L(k)} \varphi^{-1}((C^{-1})_{a, ta_1})\varphi^{-1}(C_{ta_1, ra_2})\pi_{ra_2}i \\ &= c_k^{-1} \sum_{a_1, a_2 \in_k T, r, t \in L(k)} \varphi^{-1}((C^{-1})_{a, ta_1}C_{ta_1, ra_2})\pi_{ra_2}i \\ &= c_k^{-1}\pi_{e_k a}i\end{aligned}$$

Therefore $\overline{N}(a^*)J_2 = \overline{\widehat{N}}(\widehat{\varphi}(a^*))\psi_k J_2$. For J_3 we have $\overline{N}(a^*)J_3 = c_k^{-1}\pi_{e_k a}j'\sigma_2$ and

$$\begin{aligned}\overline{\widehat{N}}(\widehat{\varphi}(a^*))\psi_k J_3 &= \sum_{a_1 \in_k T, t \in L(k)} (C^{-1})_{a, ta_1} * \overline{\widehat{N}}(a_1^*)\overline{J}_3 t^{-1}\overline{\Pi}_3\psi_k J_3 \\ &= c_k^{-1} \sum_{a_1 \in_k T, t \in L(k)} \varphi^{-1}((C^{-1})_{a, ta_1})\pi_{ta_1}\overline{j}\overline{\sigma}_2\overline{C} \\ &= c_k^{-1} \sum_{a_1 \in_k T, t \in L(k)} \varphi^{-1}((C^{-1})_{a, ta_1})\pi_{ta_1}Cj'\sigma_2 \\ &= c_k^{-1} \sum_{a_1, a_2 \in_k T, t, r \in L(k)} \varphi^{-1}((C^{-1})_{a, ta_1})\pi_{ta_1}C\xi_{ra_2}\pi_{ra_2}j'\sigma_2 \\ &= c_k^{-1} \sum_{a_1, a_2 \in_k T, t, r \in L(k)} \varphi^{-1}((C^{-1})_{a, ta_1})\varphi^{-1}(C_{ta_1, ra_2})\pi_{ra_2}j'\sigma_2 \\ &= c_k^{-1}\pi_{e_k a}j'\sigma_2\end{aligned}$$

Thus $\overline{N}(a^*)J_3 = \overline{\widehat{N}}(\widehat{\varphi}(a^*))\psi_k J_3$.

Finally, $\overline{N}(a^*)J_4 = 0$ and:

$$\overline{\widehat{N}}(\varphi(a^*))\psi_k J_4 = \sum_{a_1 \in_k T, t \in L(k)} (C^{-1})_{a, ta_1} * \overline{\widehat{N}}(a_1^*) \overline{J}_4 t^{-1} \overline{\Pi}_4 \psi_k J_4 = 0.$$

and 4.8 holds.

Suppose now that $z =^* b$ where $b \in T_k$. In this case $\tau(z) = \sigma(b) \neq k$ and $\sigma(z) = \tau(b) = k$. By [2, Proposition 8.11]:

$$\widehat{\varphi}(^*b) = \sum_{r \in L(k), b_1 \in T_k} r^{-1}(^*b_1)(D^{-1})_{b_1 r, b}$$

We have to show that

$$(4.9) \quad \psi_k \overline{N}(^*b) = \overline{\widehat{N}}(\widehat{\varphi}(^*b))$$

On one hand, $\overline{\Pi}_1 \psi_k \overline{N}(^*b) = \overline{\Pi}_1 \psi_k J_1 \Pi_1 \overline{N}(^*b) = -\underline{D}^{-1} \pi_1 p \xi_{be_k}$. On the other hand:

$$\begin{aligned} \overline{\Pi}_1 \overline{\widehat{N}}(\widehat{\varphi}(^*b)) &= \sum_{b_1 \in T_k, r \in L(k)} r^{-1} \overline{\Pi}_1 \overline{\widehat{N}}(^*b_1) \varphi^{-1}((D^{-1})_{b_1 r, b}) \\ &= - \sum_{r \in L(k), b_1 \in T_k} r^{-1} \overline{\pi}_1 \overline{p} \xi_{b_1 e_k} \varphi^{-1}((D^{-1})_{b_1 r, b}) \\ &= - \sum_{r \in L(k), b_1 \in T_k} \overline{\pi}_1 \overline{p} \xi_{b_1 r} \pi_{b_1 r} D^{-1} \xi_{be_k} \\ &= -\overline{\pi}_1 \overline{p} D^{-1} \xi_{be_k} \\ &= -\overline{\pi}_1 D_1^{-1} p D D^{-1} \xi_{be_k} \\ &= -\overline{\pi}_1 D_1^{-1} p \xi_{be_k} \\ &= -\underline{D}^{-1} \pi_1 p \xi_{be_k} \end{aligned}$$

Consequently

$$\overline{\Pi}_1 \psi_k \overline{N}(^*b) = \overline{\Pi}_1 \overline{\widehat{N}}(\widehat{\varphi}(^*b))$$

Now consider the map

$$\widehat{\gamma}' : N_{out} \rightarrow \text{im}(\widehat{\gamma})$$

where $\widehat{\gamma} = \widehat{i\gamma'}$. Let us consider $\overline{\Pi}_2$. We have:

$$\overline{\Pi}_2 \psi_k \overline{N}(^*b) = \overline{\Pi}_2 \psi_k J_2 \Pi_2 \overline{N}(^*b) = -C_1 \gamma' \xi_{be_k}$$

and

$$\begin{aligned}
\overline{\Pi}_2 \widehat{N}(\widehat{\varphi}(*b)) &= \sum_{r \in L(k), b_1 \in T_k} r^{-1} \overline{\Pi}_2 \widehat{N}(*b_1) \varphi^{-1}((D^{-1})_{b_1 r, b}) \\
&= - \sum_{r \in L(k), b_1 \in T_k} r^{-1} \widehat{\gamma}' \xi_{b_1 e_k} \varphi^{-1}((D^{-1})_{b_1 r, b}) \\
&= - \sum_{r \in L(k), b_1 \in T_k} \widehat{\gamma}' r^{-1} \xi_{b_1 e_k} \varphi^{-1}((D^{-1})_{b_1 r, b}) \\
&= - \sum_{r \in L(k), b_1 \in T_k} \widehat{\gamma}' \xi_{b_1 r} \pi_{b_1 r} D^{-1} \xi_{be_k} \\
&= - \widehat{\gamma}' D^{-1} \xi_{be_k}
\end{aligned}$$

Also:

$$\bar{i} C_1 \gamma' \xi_{be_k} = C i \gamma' \xi_{be_k} = C \gamma \xi_{be_k} = \hat{\gamma} D^{-1} \xi_{be_k} = \bar{i} \widehat{\gamma}' D^{-1} \xi_{be_k}$$

Since \bar{i} is injective, then $C_1 \gamma' \xi_{be_k} = \widehat{\gamma}' D^{-1} \xi_{be_k}$. Therefore

$$\overline{\Pi}_2 \psi_k \overline{N}(*b) = \overline{\Pi}_2 \widehat{N}(\widehat{\varphi}(*b))$$

Finally:

$$\begin{aligned}
\overline{\Pi}_3 \psi_k \overline{N}(*b) &= \overline{\Pi}_3 \psi_k J_3 \Pi_3 \overline{N}(*b) = 0 \\
\overline{\Pi}_3 \widehat{N}(\widehat{\varphi}(*b)) &= \sum_{r \in L(k), b_1 \in T_k} r^{-1} \overline{\Pi}_3 \widehat{N}(*b_1) \varphi^{-1}((D^{-1})_{b_1 r, b}) = 0 \\
\overline{\Pi}_4 \psi_k \overline{N}(*b) &= \overline{\Pi}_4 \psi_k J_4 \Pi_4 \overline{N}(*b) = 0 \\
\overline{\Pi}_4 \widehat{N}(\widehat{\varphi}(*b)) &= \sum_{r \in L(k), b_1 \in T_k} r^{-1} \overline{\Pi}_4 \widehat{N}(*b_1) \varphi^{-1}((D^{-1})_{b_1 r, b}) = 0
\end{aligned}$$

and the proof of Proposition 5 is now complete. \square

Theorem 1. Let $\varphi : \mathcal{F}_S(M_1) \rightarrow \mathcal{F}_S(M)$ be an algebra isomorphism with $\varphi|_S = id_S$ and let $\mathcal{N} = (N, V)$ be a decorated representation of $(\mathcal{F}_S(M_1), P)$ where P is a potential such that $e_k P e_k = 0$. Then there exists an algebra isomorphism:

$$\widehat{\varphi} : \mathcal{F}_S(\mu_k M_1) \rightarrow \mathcal{F}_S(\mu_k M)$$

satisfying the following conditions:

- (a) $\widehat{\varphi}(\mu_k P)$ is cyclically equivalent to $\mu_k(\varphi(P))$.
- (b) There exists an isomorphism of decorated representations $\psi : \widetilde{\mu}_k(\mathcal{N}) \rightarrow \widetilde{\mu}_k(\widehat{\mathcal{N}})$.

Proof. Part (a) follows by [2, Theorem 8.14]. Note that φ is equal to the composition of an algebra isomorphism $\mathcal{F}_S(M_1) \rightarrow \mathcal{F}_S(M)$, induced by an isomorphism of S -bimodules $M_1 \rightarrow M$, with a unitriangular automorphism of $\mathcal{F}_S(M)$. In view of Proposition 5, it suffices to prove the statement when φ is induced by an isomorphism of S -bimodules $\phi : M_1 \rightarrow M$. Let T_1 be a Z -free generating set of M_1 and let T be a Z -free generating set of M . Associated to the representation \mathcal{N} we have the following left D_k -spaces:

$$\begin{aligned}
N_{in}^1 &= \bigoplus_{a \in {}_k T_1} D_k \otimes_F N_{\tau(a)} \\
N_{out}^1 &= \bigoplus_{b \in (T_1)_k} D_k \otimes_F N_{\sigma(b)}
\end{aligned}$$

and associated to the representation $\widehat{\mathcal{N}}$ we also have the following left D_k -spaces:

$$N_{in} = \bigoplus_{a \in {}_k T} D_k \otimes_F N_{\tau(a)}$$

$$N_{out} = \bigoplus_{b \in T_k} D_k \otimes_F N_{\sigma(b)}$$

Let $\gamma_1 : N_{out}^1 \rightarrow N_{in}^1$, $\alpha_1 : N_{in}^1 \rightarrow N_k$, $\beta_1 : N_k \rightarrow N_{out}^1$ be the maps associated to \mathcal{N} and $\gamma : N_{out} \rightarrow N_{in}$, $\alpha : N_{in} \rightarrow N_k$ and $\beta : N_k \rightarrow N_{out}$ be the maps associated to $\widehat{\mathcal{N}}$.

For $a_1 \in {}_k T_1, r \in L(k)$ we have:

$$\phi(ra_1) = \sum_{r' \in L(k), a \in {}_k T} r' a C_{r'a, ra_1}$$

For $b_1 \in (T_1)_k, t \in L(k)$ we have:

$$\phi(b_1 t) = \sum_{b \in T_k, t' \in L(k)} D_{b_1 t, bt'} b t'$$

where $C_{r'a, ra_1}, D_{b_1 t, bt'} \in S$. Define $C : N_{in}^1 \rightarrow N_{in}$ as the linear map such that for all $r, r' \in L(k)$, $a, a_1 \in {}_k T_1$, the map:

$$\pi_{r'a} C \xi_{ra_1} : N_{\tau(a_1)} \rightarrow N_{\tau(a)}$$

is given by $\pi_{r'a} C \xi_{ra_1}(n) = C_{r'a, ra_1} n$. Similarly, we define $D : N_{out} \rightarrow N_{out}^1$ as the linear map such that for all $t, t' \in L(k)$, $b, b_1 \in T_k$, the map:

$$\pi_{b_1 t} D \xi_{bt'} : N_{\sigma(b)} \rightarrow N_{\sigma(b_1)}$$

is given by $\pi_{b_1 t} D \xi_{bt'}(n) = D_{b_1 t, bt'} n$.

Let us show that $\alpha C = \alpha_1$, $D\beta = \beta_1$. If $n \in N_{\tau(a_1)}$, then

$$\begin{aligned} \alpha C \xi_{ra_1}(n) &= \sum_{r'a' \in {}_k \hat{T}_1} \alpha \xi_{r'a'} \pi_{r'a'} C \xi_{ra_1}(n) \\ &= \sum_{r'a' \in {}_k \hat{T}_1} \phi^{-1}(r'a' C_{r'a', ra_1}) n = \phi^{-1} \phi(ra_1) n \\ &= ra_1 n = \alpha_1 \xi_{ra_1}(n) \end{aligned}$$

whence $\alpha C = \alpha_1$. If $n \in N_{\sigma(b)}$, $b_1 \in (T_1)_k$ and $t \in L(k)$, then

$$\begin{aligned} \pi_{b_1 t} D \beta(n) &= \sum_{bt' \in \hat{T}_k} \pi_{b_1 t} D \xi_{bt'} \pi_{bt'} \beta(n) \\ &= \sum_{bt' \in \hat{T}_k} D_{b_1 t, bt'} (\phi^{-1}(bt') n) \\ &= \phi^{-1} \left(\sum_{bt' \in \hat{T}_k} D_{b_1 t, bt'} bt' n \right) \\ &= \phi^{-1} \phi(b_1 t) n \\ &= b_1 t n = \pi_{b_1 t} \beta_1(n) \end{aligned}$$

hence $D\beta = \beta_1$.

We now show that $\hat{\gamma} = C\gamma_1 D$. Let $br \in \hat{T}_k$, $sa \in_k \hat{T}$, $n \in N_{\sigma(b)}$. We have:

$$\hat{\gamma}_{sa,br}(n) = \sum_{w \in L(k)} s^*(r^{-1}w)\varphi^{-1}(Y_{[bwa]}(\varphi(P)))n$$

Also:

$$\varphi^{-1}(Y_{[bwa]}(\varphi(P))) = \varphi^{-1}\rho^{-1}(X_{[bwa]}(\rho\varphi(P))) = \rho^{-1}\hat{\varphi}^{-1}(X_{[bwa]}(\hat{\varphi}\rho(P)))$$

Let

$$T_1^\mu = \{\rho(c_1) : c_1 \in T_1 \cap \bar{e}_k M_1 \bar{e}_k\} \cup \{a_1^* : a_1 \in_k T_1\} \cup \{^*b_1 : b_1 \in (T_1)_k\} \cup \{\rho(b_1 r a_1) : b_1 \in (T_1)_k, a_1 \in_k T_1, r \in L(k)\}$$

denote the Z -free generating set of $\mu_k M_1$ associated to T_1 . Likewise, let T^μ denote the Z -free generating set of $\mu_k M$ associated to T . Then:

$$X_{[bwa]}(\hat{\varphi}(\rho(P))) = \sum_{s \in L(\sigma(b))} \sum_{c \in (T_1)_{k,\hat{k}}^\mu} (s\rho(bwa))^*(\hat{\Delta}(\hat{\varphi}(c))\diamond z(c))s$$

where $z(c) = \hat{\varphi}(X_c(\rho(P)))$.

The elements of the set $(T_1)_{k,\hat{k}}^\mu$ are either elements of the form $\rho(c_1)$ where $c_1 \in T_1 \cap \bar{e}_k M \bar{e}_k$ or elements of the form $\rho(b_1 r a_1)$ where $b_1 \in (T_1)_k, a_1 \in_k T_1, r \in L(k)$. We now compute $W(c_1)$ when $c_1 \in T_1 \cap \bar{e}_k M \bar{e}_k$.

$$W(c_1) = \sum_{s \in L(\sigma(b))} (s\rho(bwa))^*(\hat{\Delta}(\hat{\varphi}\rho(c_1))\diamond z(\rho(c_1)))s$$

Then $\hat{\varphi}\rho(c_1) = \rho(\varphi(c_1)) = \rho(\phi(c_1)) = \sum_i r_i \rho(c_i) t_i$ where $r_i, t_i \in S$, $c_i \in T \cap \bar{e}_k M \bar{e}_k$.

Using [2, Lemma 5.2] one gets:

$$\begin{aligned} W(c_1) &= \sum_i \sum_{s \in L(\sigma(b))} (s\rho(bwa))^*(\hat{\Delta}(\rho(c_i))\diamond t_i z(\rho(c_1))r_i) s \\ &= \sum_i \sum_{s \in L(\sigma(b))} (s\rho(bwa))^*(\rho(c_i) t_i z(\rho(c_1)) r_i) s = 0. \end{aligned}$$

Therefore

$$X_{[bwa]}(\hat{\varphi}(\rho(P))) = \sum_{s \in L(\sigma(b))} \sum_{b_1 h a_1} (s\rho(bwa))^*(\hat{\Delta}(\hat{\varphi}\rho(b_1 h a_1))\diamond z(\rho(b_1 h a_1)))s$$

and

$$\hat{\varphi}(\rho(b_1 h a_1)) = \rho\varphi(b_1 h a_1) = \rho(\phi(b_1)\phi(h a_1)) = \sum_{b', r', r'', a'', w'} (w')^*(r' r'') D_{b_1, b' r'} \rho(b' w' a'') C_{r'' a'', h a_1}$$

Once again, [2, Lemma 5.2] yields:

$$\begin{aligned} X_{[bwa]}(\hat{\varphi}(\rho(P))) &= \\ \sum_{s \in L(\sigma(b)), b_1 h a_1, b', r', r'', a'', w'} & (s\rho(bwa))^*((w')^*(r' r'') \rho(b' w' a'') C_{r'' a'', h a_1} z(\rho(b_1 h a_1)) D_{b_1, b' r'}) s \end{aligned}$$

Therefore:

$$X_{[bwa]}(\hat{\varphi}(\rho(P))) = \sum_{b_1 ha_1, r', r''} w^*(r' r'') C_{r'' a, ha_1} z(\rho(b_1 ha_1)) D_{b_1, br'}$$

Note that

$$z(\rho(b_1 ha_1)) = \hat{\varphi}(X_{[b_1 ha_1]}(\rho(P))) = \hat{\varphi}(\rho(Y_{[b_1 ha_1]}(P))) = \rho\varphi(Y_{[b_1 ha_1]}(P))$$

This yields the equality:

$$X_{[bwa]}(\hat{\varphi}(\rho(P))) = \rho\varphi \left(\sum_{b_1 ha_1, r', r''} w^*(r' r'') C_{r'' a, ha_1} Y_{[b_1 ha_1]}(P) D_{b_1, br'} \right)$$

Consequently:

$$\begin{aligned} \hat{\gamma}_{sa, br}(n) &= \sum_{w \in L(k)} s^*(r^{-1} w) \sum_{b_1 ha_1, r', r''} w^*(r' r'') C_{r'' a, ha_1} Y_{[b_1 ha_1]}(P) D_{b_1, br'} n \\ &= \sum_{b_1 ha_1, r', r''} s^*(r^{-1} r' r'') C_{r'' a, ha_1} Y_{[b_1 ha_1]}(P) D_{b_1, br'} n \end{aligned}$$

A similar argument to the proof of Lemma 4 yields that $\hat{\gamma}_{sa, br}(n)$ is equal to $(C\gamma_1 D)_{sa, br}(n)$. The desired right-equivalence is analogous to the one constructed in Proposition 5. \square

Proposition 6. *The right-equivalence class of the representation $\widetilde{\mu}_k(\mathcal{N})$ is determined by the right-equivalence class of the representation \mathcal{N} .*

Proof. Let $\mathcal{N} = (N, V)$ be a decorated representation of $(\mathcal{F}_S(M), P)$ and let $\mathcal{N}' = (N', V')$ be a decorated representation of $(\mathcal{F}_S(M'), P')$. Suppose that such representations are right-equivalent, then there exists an algebra isomorphism $\varphi : \mathcal{F}_S(M) \rightarrow \mathcal{F}_S(M')$, with $\varphi|_S = id_S$, such that $\varphi(P)$ is cyclically equivalent to P' . Recall that associated to the representation $\mathcal{N} = (N, V)$ we have a decorated representation $\widehat{\mathcal{N}} = (\widehat{N}, V)$ of $(\mathcal{F}_S(M'), \varphi(P))$ where $\widehat{N} = N$ as F -vector spaces, and for $u \in \mathcal{F}_S(M')$ and $n \in N$ we let $u * n := \varphi^{-1}(u)n$. Let $\psi : \widehat{N} \rightarrow N'$ be the isomorphism of F -vector spaces induced by the right-equivalence between \mathcal{N} and \mathcal{N}' . Note that ψ is an isomorphism of left $\mathcal{F}_S(M')$ -modules because if $u \in \mathcal{F}_S(M')$ and $n \in \widehat{N}$ then $\psi(un) = \psi(\varphi^{-1}(u)n) = \varphi(\varphi^{-1}(u))\psi(n) = u\psi(n)$. Now, let us show that:

- (a) $\widetilde{\mu}_k(\mathcal{N})$ is right-equivalent to $\widetilde{\mu}_k(\widehat{\mathcal{N}})$
- (b) $\widetilde{\mu}_k(\widehat{\mathcal{N}})$ is right-equivalent to $\widetilde{\mu}_k(\mathcal{N}')$

Note that (a) follows immediately by Theorem 1. We now prove (b). Consider the cyclically equivalent potentials $\varphi(P)$ and P' in $\mathcal{F}_S(M')$. By [2, Proposition 8.15] we have that $\mu_k(\varphi(P))$ is cyclically equivalent to $\mu_k P'$, in particular $\mu_k(\varphi(P))$ is right-equivalent to $\mu_k P'$ via the identity map $id_{\mathcal{F}_S(\mu_k M')}$.

Note that ψ induces isomorphisms of F -vector spaces:

$$\begin{aligned} \hat{\psi}_i &: N_{in} \rightarrow N'_{in} \\ \hat{\psi}_o &: N_{out} \rightarrow N'_{out} \end{aligned}$$

Let $\rho = \psi^{-1} : N' \rightarrow \widehat{N}$, then ρ also induces isomorphisms of F -vector spaces:

$$\begin{aligned} \rho_i &: N'_{in} \rightarrow N_{in} \\ \rho_o &: N'_{out} \rightarrow N_{out} \end{aligned}$$

Let $\psi_k : N_k \rightarrow N'_k$ and $\rho_k : N'_k \rightarrow N_k$ be the maps induced by ψ and ρ . Then we have the following equalities:

$$\begin{aligned}
 \psi_k \alpha &= \alpha' \hat{\psi}_i \\
 \beta' \psi_k &= \hat{\psi}_o \beta \\
 \hat{\psi}_i \gamma &= \gamma' \hat{\psi}_o \\
 \rho_k \alpha' &= \alpha \rho_i \\
 \beta \rho_k &= \rho_o \beta' \\
 \rho_i \gamma' &= \gamma \rho_o
 \end{aligned}
 \tag{4.10}$$

Observe that $\hat{\psi}_i$ induces a map $\ker(\alpha) \rightarrow \ker(\alpha')$ and ρ_i induces a map $\ker(\alpha') \rightarrow \ker(\alpha)$ such that $\rho_i = (\hat{\psi}_i)^{-1}$. Hence $\hat{\psi}_i$ induces an isomorphism between $\ker(\alpha)$ and $\ker(\alpha')$. Similarly, $\hat{\psi}_i$ induces an isomorphism $\text{im}(\gamma) \rightarrow \text{im}(\gamma')$, $\hat{\psi}_o$ induces an isomorphism $\ker(\gamma) \rightarrow \ker(\gamma')$ and $\hat{\psi}_o$ induces an isomorphism $\text{im}(\beta) \rightarrow \text{im}(\beta')$.

To construct $\overline{N'}$ we choose the splitting data (p', σ') in terms of the splitting data (p, σ) of \overline{N} as follows:

$$\begin{aligned}
 p' &= \psi_1 p (\hat{\psi}_o)^{-1} \\
 \sigma' &= \psi_2 \sigma (\overline{\psi}_2)^{-1}
 \end{aligned}
 \tag{4.11}$$

where ψ_1 is the restriction of $\hat{\psi}_o$ to $\ker(\gamma)$, ψ_2 is the restriction of the isomorphism $\hat{\psi}_i : N_{in} \rightarrow N'_{in}$ to $\ker(\alpha)$ and $\overline{\psi}_2 : \frac{\ker(\alpha)}{\text{im}(\gamma)} \rightarrow \frac{\ker(\alpha')}{\text{im}(\gamma')}$ is the isomorphism induced by $\hat{\psi}_i$.

Define $\tilde{\psi} : \overline{N} \rightarrow \overline{N'}$ as the map $\psi : \widehat{N}_i \rightarrow N'_i$ for all $i \neq k$ and if $i = k$ then $\tilde{\psi}$ is the map given in diagonal matrix form by the maps previously defined. Let $d_1 \in D_k$, $b \in T_k$ and $d_2 \in D_{\sigma(b)}$. Let us show the following equality holds for every $n \in N_{\sigma(b)}$:

$$\tilde{\psi}(d_1(*b)d_2 \cdot n) = d_1(*b)d_2 \cdot \tilde{\psi}(n)$$

First consider the term on the left-hand side. We have

$$\begin{aligned}
 \tilde{\psi}(d_1(*b)d_2 \cdot n) &= \tilde{\psi}(d_1 \overline{N}(*b)(d_2 n)) \\
 &= \tilde{\psi} \begin{pmatrix} -d_1 \pi_1 p \xi_{be_k}(d_2 n) \\ -d_1 \gamma' \xi_{be_k}(d_2 n) \\ 0 \\ 0 \end{pmatrix}
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 d_1(*b)d_2 \cdot \tilde{\psi}(n) &= d_1(*b)d_2 \cdot \psi(n) \\
 &= \begin{pmatrix} -d_1 \pi'_1 p' \xi_{be_k}(d_2 \psi(n)) \\ -d_1 \gamma' \xi_{be_k}(d_2 \psi(n)) \\ 0 \\ 0 \end{pmatrix}
 \end{aligned}$$

In what follows, we will use the notation of 4.2 for the projection maps associated to $\overline{N'_k}$.

We now show that $\Pi'_1(\tilde{\psi}(d_1\overline{N}(*b)(d_2n))) = \Pi'_1(d_1(*b)d_2 \cdot \psi(n))$.

On one hand

$$\Pi'_1(\tilde{\psi}(d_1\overline{N}(*b)(d_2n))) = \pi'_1(-d_1\psi_1p\xi_{be_k}(d_2n))$$

and on the other hand

$$\begin{aligned}\Pi'_1(d_1(*b)d_2 \cdot \psi(n)) &= -d_1\pi'_1p'\xi_{be_k}(d_2\psi(n)) \\ &= -d_1\pi'_1p'\hat{\psi}_o\xi_{be_k}(d_2n)\end{aligned}$$

By 4.11 we have $p'\hat{\psi}_o = \psi_1p$ and thus

$$-d_1\pi'_1p'\hat{\psi}_o\xi_{be_k}(d_2n) = -d_1\pi'_1\psi_1p\xi_{be_k}(d_2n) = \pi'_1(-d_1\psi_1p\xi_{be_k}(d_2n))$$

Therefore $\Pi'_1(\tilde{\psi}(d_1\overline{N}(*b)(d_2n))) = \Pi'_1(d_1(*b)d_2 \cdot \psi(n))$, as was to be shown.

We now verify that $\Pi'_2(\tilde{\psi}(d_1\overline{N}(*b)(d_2n))) = \Pi'_2(d_1(*b)d_2 \cdot \psi(n))$. We have

$$\begin{aligned}\Pi'_2(\tilde{\psi}(d_1\overline{N}(*b)(d_2n))) &= -d_1\hat{\psi}_i\gamma'\xi_{be_k}(d_2n) \\ &= -d_1\gamma'\hat{\psi}_o\xi_{be_k}(d_2n) \\ &= -d_1\gamma'\xi_{be_k}(d_2\tilde{\psi}(n)) \\ &= \Pi'_2(d_1(*b)d_2 \cdot \psi(n))\end{aligned}$$

Let $a \in_k T$, $d_1 \in D_{\tau(a)}$, $d_2 \in D_k$ and $w \in \overline{N}_k$. We now prove the following equality:

$$\tilde{\psi}(d_1a^*d_2 \cdot w) = d_1a^*d_2\tilde{\psi}(w)$$

Using 4.3 we obtain

$$\begin{aligned}\tilde{\psi}(d_1a^*d_2w) &= \tilde{\psi}(d_1c_k^{-1}\pi_{e_ka}i\Pi_2(d_2w) + d_1c_k^{-1}\pi_{e_ka}j'\sigma_2\Pi_3(d_2w)) \\ &= d_1c_k^{-1}\psi(\pi_{e_ka}i\Pi_2(d_2w)) + d_1c_k^{-1}\psi(\pi_{e_ka}j'\sigma_2\Pi_3(d_2w))\end{aligned}$$

On the other hand

$$\begin{aligned}d_1a^*d_2\tilde{\psi}(w) &= d_1\overline{N}'(a^*)(d_2\tilde{\psi}(w)) \\ &= d_1c_k^{-1}\pi'_{e_ka}i\hat{\psi}_i\Pi_2(d_2w) + d_1c_k^{-1}\pi_{e_ka}j'\sigma'_2\overline{\psi}_2(\Pi_3(d_2w))\end{aligned}$$

and thus (b) follows. This completes the proof of Proposition 6. □

5. MUTATION OF DECORATED REPRESENTATIONS

Let $(\mathcal{F}_S(M), P)$ be an algebra with potential such that $P^{(2)}$ is splittable. By [2, Theorem 7.15] there exists a decomposition of S -bimodules $M = M_1 \oplus \Xi_2(P)$ and a unitriangular automorphism $\varphi : \mathcal{F}_S(M) \rightarrow \mathcal{F}_S(M)$ such that $\varphi(P)$ is cyclically equivalent to $Q^{\geq 3} \oplus Q^{(2)}$ where $Q^{\geq 3}$ is a reduced potential in $\mathcal{F}_S(M_1)$ and $Q^{(2)}$ is a trivial potential in $\mathcal{F}_S(\Xi_2(P))$.

We have that $\widehat{\mathcal{N}}$ is a decorated representation of $(\mathcal{F}_S(M), Q^{\geq 3} \oplus Q^{(2)})$. Therefore, $\widehat{\mathcal{N}}|_{\mathcal{F}_S(M_1)}$ is a decorated representation of $(\mathcal{F}_S(M_1), Q^{\geq 3})$.

Definition 12. We will refer to the decorated representation $\widehat{\mathcal{N}}|_{\mathcal{F}_S(M_1)}$ as the reduced part of \mathcal{N} and will be denoted by \mathcal{N}_{red} .

Let k be a fixed integer in $[1, n]$ and let $M = M_1 \oplus M_2$ be a Z -freely generated S -bimodule such that for every i , $e_i M e_k \neq 0$ implies $e_k M e_i = 0$ and likewise $e_k M e_i \neq 0$ implies $e_i M e_k = 0$.

Let T_1 be a Z -free generating set of M_1 and let T_2 be a Z -free generating set of M_2 . Let P_1 be a potential in $\mathcal{F}_S(M_1)$ and let $W = \sum_{i=1}^l c_i d_i$ be a trivial potential in $\mathcal{F}_S(M_2)$, where $\{c_1, \dots, c_l, d_1, \dots, d_l\} = T_2$.

Suppose that $e_k P_1 e_k = 0$ and consider the potential $P = P_1 + W$ in $\mathcal{F}_S(M_1 \oplus M_2)$. Since $e_k P e_k = 0$, then we can consider

$$\mu_k(P) = \mu_k(P_1) + W$$

Note that

$$\mu_k M = \mu_k M_1 \oplus M_2$$

Let $\mathcal{N} = (N, V)$ be a decorated representation of $(\mathcal{F}_S(M), P)$. Recall that $\tilde{\mu}_k(\mathcal{N})|_{\mathcal{F}_S(\mu_k M_1)}$ is a decorated representation of $(\mathcal{F}_S(\mu_k M_1), \mu_k P_1)$. Recall also that $\mathcal{N}|_{\mathcal{F}_S(M_1)}$ is a decorated representation of $(\mathcal{F}_S(M_1), P_1)$.

Lemma 9. *The following equality holds: $\tilde{\mu}_k(\mathcal{N}|_{\mathcal{F}_S(M_1)}) = (\tilde{\mu}_k(\mathcal{N}))|_{\mathcal{F}_S(\mu_k M_1)}$.*

Proof. Let T_1 be a Z -free generating set of $(M_1)_0$ and let T_2 be a Z -free generating set of $(M_2)_0$. Then

$$W = \sum_{s=1}^l c_s d_s$$

where $T_2 = \{c_1, \dots, c_l, d_1, \dots, d_l\}$. Note that $e_k T_2 = T_2 e_k = 0$. We have

$$\mathcal{N}|_{\mathcal{F}_S(M_1)} = (N|_{\mathcal{F}_S(M_1)}, V)$$

Define $N' = N|_{\mathcal{F}_S(M_1)}$. For $i \neq k$, $\overline{N'}_i = N'_i = N_i = \overline{N}_i$. On the other hand

$$N'_{out} = \bigoplus_{b \in (T_1)_k} D_k \otimes_F N_{\sigma(b)} = \bigoplus_{b \in T_k} D_k \otimes_F N_{\sigma(b)} = N_{out}$$

Similarly, $N'_{in} = N_{in}$. Now let $\alpha' : N'_{in} \rightarrow N'_k$, $\beta' : N'_k \rightarrow N'_{out}$ and $\gamma' : N'_{out} \rightarrow N'_{in}$ be the maps associated to the representation $\overline{N'}$. Clearly $\alpha = \alpha'$, $\beta = \beta'$. Also:

$$Y_{[bra]}(P_1 + W) = Y_{[bra]}(P_1)$$

and thus $\gamma = \gamma'$. Then $\overline{N'}_k = \overline{N}_k$ and it follows that $\overline{N'} = \overline{N}$ as left S -modules. Since the action of $\mathcal{F}_S(\mu_k M_1)$ in $\overline{N'}$ and \overline{N} is the same, then the claim follows. \square

Proposition 7. *Let $(\mathcal{F}_S(M), P)$ be an algebra with potential where $P^{(2)}$ is splittable. Suppose that $e_k P e_k = 0$ and that $(M \otimes_S e_k M)_{cyc} = 0$. For every decorated representation \mathcal{N} of $(\mathcal{F}_S(M), P)$, the decorated representation $\tilde{\mu}_k(\mathcal{N})_{red}$ is right-equivalent to $\tilde{\mu}_k(\mathcal{N}_{red})_{red}$.*

Proof. By [2, Theorem 7.15] there exists a unitriangular automorphism:

$$\varphi_1 : \mathcal{F}_S(M) \rightarrow \mathcal{F}_S(M)$$

and a decomposition of S -bimodules $M = M_1 \oplus \Xi_2(P)$ such that $\varphi_1(P)$ is cyclically equivalent to $Q + W_1$ where Q is a reduced potential in $\mathcal{F}_S(M_1)$ and W_1 is a trivial potential in $\mathcal{F}_S(\Xi_2(P))$. Then $\mu_k(\varphi_1(P))$ is cyclically equivalent to $\mu_k(Q) + W_1$. By [2, Theorem 8.12] there exists a unitriangular automorphism:

$$\varphi_2 : \mathcal{F}_S(\mu_k M_1) \rightarrow \mathcal{F}_S(\mu_k M_1)$$

and a decomposition of S -bimodules:

$$\mu_k M_1 = M_2 \oplus \Xi_2(\mu_k(Q))$$

such that $\varphi_2(\mu_k Q) = Q_1 + W_2$ where Q_1 is a reduced potential in $\mathcal{F}_S(M_2)$ and W_2 is a trivial potential in $\mathcal{F}_S(\Xi_2(\mu_k(Q)))$. Let $\hat{\varphi}_2$ be the unitriangular automorphism of $\mathcal{F}_S(\mu_k M)$ extending φ_2 and such that $\hat{\varphi}_2$ is equal to the identity in $\mathcal{F}_S(\Xi_2(P))$. Now consider the algebra automorphism $\hat{\varphi}_1$ of $\mathcal{F}_S(\mu_k M)$ given by Theorem 1. Then

- (a) $\hat{\varphi}_1(\mu_k P)$ is cyclically equivalent to $\mu_k(\varphi_1(P))$.
- (b) There exists an isomorphism of decorated representations $\widetilde{\mu}_k(\mathcal{N}) \rightarrow \widetilde{\mu}_k(\varphi_1 \mathcal{N})$

where $\varphi_1 \mathcal{N}$ denotes the representation $\widehat{\mathcal{N}}$ (to emphasize the dependance of the action on φ_1). We have that $\hat{\varphi}_2 \hat{\varphi}_1(\mu_k P)$ is cyclically equivalent to $\hat{\varphi}_2 \mu_k(\varphi_1(P))$ and the latter is cyclically equivalent to $\hat{\varphi}_2(\mu_k Q + W_1) = \varphi_2(\mu_k(Q)) + W_1$; also, the latter potential is cyclically equivalent to $Q_1 + W_1 + W_2$ and

$$\mu_k M = M_2 \oplus \Xi_2(\mu_k Q) \oplus \Xi_2(P)$$

with Q_1 a reduced potential in $\mathcal{F}_S(M_2)$ and $W_1 + W_2$ a trivial potential in $\mathcal{F}_S(\Xi_2(\mu_k Q) \oplus \Xi_2(P))$. Then $(\widetilde{\mu}_k(\mathcal{N}))_{red}$ is isomorphic to $\hat{\varphi}_2(\widetilde{\mu}_k(\varphi_1 \mathcal{N}))|_{\mathcal{F}_S(M_2)}$. Note that, by Lemma 9, $\hat{\varphi}_2 \widetilde{\mu}_k(\varphi_1 \mathcal{N})|_{\mathcal{F}_S(\mu_k M_1)}$ is isomorphic to $\hat{\varphi}_2 \widetilde{\mu}_k(\varphi_1 \mathcal{N}|_{\mathcal{F}_S(M_1)})$. The claim follows. \square

Definition 13. Let (N, V) be a decorated representation of the algebra with potential $(\mathcal{F}_S(M), P)$ where P is reduced. We define the mutation of the decorated representation \mathcal{N} in k as:

$$\mu_k(\mathcal{N}) := \widetilde{\mu}_k(\mathcal{N})_{red}$$

Corollary 1. The correspondence $\mathcal{N} \mapsto \mu_k(\mathcal{N})$ is a well-defined transformation on the set of right-equivalence classes of decorated representations of reduced algebras with potentials.

Theorem 2. The mutation μ_k of decorated representations is an involution; that is, for every decorated representation \mathcal{N} of a reduced algebra with potential $(\mathcal{F}_S(M), P)$, the decorated representation $\mu_k^2(\mathcal{N})$ is right-equivalent to \mathcal{N} .

Proof. Let $\mathcal{N} = (N, V)$ be a decorated representation of $(\mathcal{F}_S(M), P)$. First note that

$$\begin{aligned} \mu_k^2(\mathcal{N}) &= \mu_k(\mu_k(\mathcal{N})) \\ &= \widetilde{\mu}_k(\mu_k(\mathcal{N}))_{red} \\ &= \widetilde{\mu}_k(\widetilde{\mu}_k(\mathcal{N})_{red})_{red} \end{aligned}$$

In view of Proposition 7, $\widetilde{\mu}_k(\widetilde{\mu}_k(\mathcal{N})_{red})_{red}$ is right-equivalent to $\widetilde{\mu}_k(\widetilde{\mu}_k(\mathcal{N}))_{red} = \widetilde{\mu}_k^2(\mathcal{N})_{red}$. Thus, it suffices to show that $\widetilde{\mu}_k^2(\mathcal{N})_{red}$ is right-equivalent to \mathcal{N} . We write the representation $\widetilde{\mu}_k^2(\mathcal{N})$ as $(\overline{N}, \overline{V})$ and this representation is associated to the algebra with potential

$(\mathcal{F}_S(\mu_k^2 M), \mu_k^2 P)$. By [2, Proposition 8.8] and [2, Theorem 8.17] we have:

$$\begin{aligned}\mu_k^2 M &= M \oplus Me_k M \oplus M^* e_k (^* M) \\ \mu_k^2 P &= \rho(P) + \sum_{bt, sa} ([btsa][(sa)^* (^* bt)] + [(sa)^* (^* (bt))](bt)(sa))\end{aligned}$$

By [2, Theorem 8.17] there exists an algebra automorphism $\psi : \mathcal{F}_S(\mu_k^2 M) \rightarrow \mathcal{F}_S(\mu_k^2 M)$ such that $\psi(\mu_k^2 P)$ is cyclically equivalent to $P \oplus W$ where W is a trivial potential in $\mathcal{F}_S(Me_k M \oplus M^* e_k (^* M))$. Such automorphism ψ restricts to an automorphism $\psi_0 : \mathcal{F}_S(M) \rightarrow \mathcal{F}_S(M)$ such that $\psi_0(b) = -b$ for every $b \in T_k$ and $\psi_0(x) = x$ for every $x \in T \setminus T_k$. In view of Definition 13, the representation $\widetilde{\mu}_k(\mathcal{N})_{red}$ can be realized as $(\overline{\overline{N}}, \overline{\overline{V}})$, the latter being associated to the algebra with potential $(\mathcal{F}_S(M), P)$, and the action of $\mathcal{F}_S(M)$ in $\overline{\overline{N}}$ is given by $u \cdot n = \psi_0^{-1}(u)n$.

Let us show that $\overline{\alpha} : N_{out} = \overline{\overline{N}}_{in} \longrightarrow \overline{\overline{N}}_k$ is the F -linear map such that for each $b \in T_k$ and $r \in L(k)$, $\overline{\alpha}\xi_{br} = r^{-1}\overline{\overline{N}}(^*b)$. We have

$$\begin{aligned}\overline{\alpha}\xi_{br}(n) &= \sum_{w \in L(k)} \overline{\alpha}\xi_{w(^*b)}\pi_{w(^*b)}\xi_{br}(n) \\ &= \sum_{w \in L(k)} \overline{\alpha}\xi_{w(^*b)}\pi'_{w(^*b)}(r^{-1} \otimes n) \\ &= \sum_{w \in L(k)} \overline{\alpha}\xi_{w(^*b)}w^*(r^{-1})n \\ &= \sum_{w \in L(k)} \overline{\overline{N}}(w(^*b))(w^*(r^{-1})n) \\ &= \overline{\overline{N}}\left(\sum_{w \in L(k)} w^*(r^{-1})w(^*b)\right)(n) \\ &= \overline{\overline{N}}(r^{-1}(^*b))(n) \\ &= r^{-1}\overline{\overline{N}}(^*b)(n)\end{aligned}$$

We now show that $\ker(\overline{\alpha}) = \text{im}(\beta)$.

Let $x \in N_{out}$, using 3.15 we have $x = \sum_{b \in T_k, r \in L(k)} \xi_{br}\pi_{br}(x)$. Therefore

$$\begin{aligned}\overline{\alpha}(x) &= \sum_{b \in T_k, r \in L(k)} \overline{\alpha}\xi_{br}\pi_{br}(x) \\ &= \sum_{b \in T_k, r \in L(k)} r^{-1}\overline{\overline{N}}(^*b)(\pi_{br}(x))\end{aligned}$$

thus $\overline{\alpha}(x) = 0$ if and only if $\Pi_i \left(\sum_{b \in T_k, r \in L(k)} r^{-1}\overline{\overline{N}}(^*b)(\pi_{br}(x)) \right) = 0$ for every $i \in \{1, 2, 3, 4\}$.

Remembering 4.4 yields

$$(5.1) \quad \Pi_1 \left(\sum_{b \in T_k, r \in L(k)} r^{-1}\overline{\overline{N}}(^*b)(\pi_{br}(x)) \right) = - \sum_{b \in T_k, r \in L(k)} r^{-1}\pi_1 p \xi_{be_k} \pi_{br}(x)$$

$$(5.2) \quad \Pi_2 \left(\sum_{b \in T_k, r \in L(k)} r^{-1} \overline{N}(*b) (\pi_{br}(x)) \right) = - \sum_{b \in T_k, r \in L(k)} r^{-1} \gamma' \xi_{be_k} \pi_{br}(x)$$

so if $\overline{\alpha}(x) = 0$ then 5.2 implies that

$$(5.3) \quad \sum_{b \in T_k, r \in L(k)} r^{-1} \xi_{be_k} \pi_{br}(x) \in \ker(\gamma)$$

On the other hand, if $\overline{\alpha}(x) = 0$ then by 5.1, $\sum_{b \in T_k, r \in L(k)} r^{-1} p \xi_{be_k} \pi_{br}(x) \in \text{im}(\beta)$. Note that

$$\sum_{b \in T_k, r \in L(k)} r^{-1} p \xi_{be_k} \pi_{br}(x) = p \left(\sum_{b \in T_k, r \in L(k)} r^{-1} \xi_{be_k} \pi_{br}(x) \right)$$

Using 5.3 and the fact that p is a left inverse of the inclusion map $j : \ker(\gamma) \rightarrow N_{out}$ yields $p \left(\sum_{b \in T_k, r \in L(k)} r^{-1} \xi_{be_k} \pi_{br}(x) \right) = \sum_{b \in T_k, r \in L(k)} r^{-1} \xi_{be_k} \pi_{br}(x)$. Finally, note that

$$\sum_{b \in T_k, r \in L(k)} r^{-1} \xi_{be_k} \pi_{br}(x) = \sum_{b \in T_k, r \in L(k)} \xi_{br} \pi_{br}(x)$$

and by 3.15 $\sum_{b \in T_k, r \in L(k)} \xi_{br} \pi_{br}(x) = x$. Therefore, $\overline{\alpha}(x) = 0$ if and only if $\sum_{b \in T_k, r \in L(k)} r^{-1} p \xi_{be_k} \pi_{br}(x) = x \in \text{im}(\beta)$. This proves that:

$$(5.4) \quad \ker(\overline{\alpha}) = \text{im}(\beta)$$

As a consequence, $\Pi_1 \overline{\alpha}(x) = -\pi_1 p$ y $\Pi_2 \overline{\alpha} = -\gamma'$. Therefore

$$(5.5) \quad \text{im}(\overline{\alpha}) = \frac{\ker(\gamma)}{\text{im}(\beta)} \oplus \text{im}(\gamma) \oplus \{0\} \oplus \{0\}$$

We now show that $\overline{\beta} : \overline{N}_k \rightarrow \overline{N}_{out} = N_{in}$ is the F -linear map such that for each $r \in L(k)$ and $a \in {}_k T$, $\pi_{ra} \overline{\beta} = \overline{N}(a^*) r^{-1}$. We have:

$$\begin{aligned} \pi_{ra} \overline{\beta}(n) &= \sum_{w \in L(k)} \pi_{ra} \xi_{a^* w} \pi_{a^* w} \overline{\beta}(n) \\ &= \sum_{w \in L(k)} \pi_{ra} \xi_{a^* w} \overline{N}(a^* w)(n) \\ &= \sum_{w \in L(k)} \pi'_{ra} (w^{-1} \otimes \overline{N}(a^* w)(n)) \\ &= \overline{N} \left(a^* \sum_{w \in L(k)} r^*(w^{-1}) w \right) (n) \end{aligned}$$

By Remark 3, $\sum_{w \in L(k)} r^*(w^{-1}) w = r^{-1}$. It follows that

$$\pi_{ra}\bar{\beta}(n) = \bar{N}(a^*r^{-1})(n) = \bar{N}(a^*)(r^{-1}n)$$

We now prove that

$$(5.6) \quad \ker(\bar{\beta}) = \frac{\ker(\gamma)}{\text{im}(\beta)} \oplus \{0\} \oplus \{0\} \oplus V_k$$

Let $w \in \bar{N}_k$, then $w = \sum_{l=1}^4 J_l \Pi_l(w)$. Suppose that $\bar{\beta}(w) = 0$, then $\pi_{ra}\bar{\beta}(w) = 0$ for every $r \in L(k)$ and $a \in_k T$. Using Lemma 2 and 4.3 we obtain the following equalities:

$$\begin{aligned} 0 &= \pi_{ra}\bar{\beta}(w) = \sum_{l=1}^4 \pi_{ra}\bar{\beta}(J_l \Pi_l(w)) \\ &= \sum_{l=1}^4 \bar{N}(a^*) (J_l r^{-1} \Pi_l(w)) \\ &= \bar{N}(a^*) J_2 (r^{-1} \Pi_2(w)) + \bar{N}(a^*) J_3 (r^{-1} \Pi_3(w)) \\ &= c_k^{-1} \pi_{e_k a} i (r^{-1} \Pi_2(w)) + c_k^{-1} \pi_{e_k a} j' \sigma_2 (r^{-1} \Pi_3(w)) \\ &= c_k^{-1} \pi_{ra} (i \Pi_2(w)) + c_k^{-1} \pi_{ra} (j' \sigma_2 \Pi_3(w)) \\ &= c_k^{-1} \pi_{ra} (i \Pi_2(w) + j' \sigma_2 \Pi_3(w)) \end{aligned}$$

Since this is true for all projections π_{ra} , then

$$(5.7) \quad 0 = i \Pi_2(w) + j' \sigma_2 \Pi_3(w) = \Pi_2(w) + \sigma_2 \Pi_3(w)$$

By 4.2, $\Pi_2(w) \in \text{im}(\gamma)$. Applying π_2 to 5.7, where $\pi_2 : \ker(\alpha) \rightarrow \frac{\ker(\alpha)}{\text{im}(\gamma)}$ is the projection map, yields $\pi_2 \sigma_2 \Pi_3(w) = 0$. Since $\pi_2 \sigma_2 = \text{id}_{\ker(\alpha)/\text{im}(\gamma)}$, then $\Pi_3(w) = 0$. Substituting $\Pi_3(w) = 0$ into 5.7 gives $\Pi_2(w) = 0$. Consequently:

$$\ker(\bar{\beta}) = \frac{\ker(\gamma)}{\text{im}(\beta)} \oplus \{0\} \oplus \{0\} \oplus V_k$$

and the proof of 5.6 is now complete. We also have

$$(5.8) \quad \text{im}(\bar{\beta}) = \ker(\alpha)$$

We now prove the following formula

$$(5.9) \quad \bar{\gamma} = c_k \beta \alpha$$

First, we compute $Y_{[a^*w(*b)]}(\mu_k P)$ where $a \in_k T, b \in T_k, w \in L(k)$ and Δ_k is the following potential

$$\Delta_k = \sum_{sa_1 \in_k \hat{T}, b_1 t \in \tilde{T}_k} [b_1 t s a_1] ((s a_1)^*) (* (b_1 t))$$

Note that Δ_k is cyclically equivalent to the following potential

$$\Delta'_k = \sum_{sa_1, b_1 t} a_1^* s^{-1} t^{-1} (* b_1) [b_1 t s a_1].$$

Therefore

$$\begin{aligned}\Delta'_k &= \sum_{sa_1, b_1} \sum_{t \in L(k)} (v_1^{-1})^* (s^{-1}t^{-1}) v^* (ts) a_1^* v_1^{-1} (*b_1) [b_1 v a_1] \\ &= \sum_{sa_1, b_1} \sum_{v, v_1, t \in L(k)} (v_1^{-1})^* (s^{-1}t^{-1}) v^* (ts) a_1^* v_1^{-1} (*b_1) [b_1 v a_1]\end{aligned}$$

By [2, Proposition 7.5] we have:

$$\sum_{t \in L(k)} v^* (ts) (v_1^{-1})^* (s^{-1}t^{-1}) = \delta_{v, v_1}$$

thus

$$\begin{aligned}\Delta'_k &= \sum_{sa_1, b_1} \sum_{v \in L(k)} a_1^* v^{-1} (*b_1) [b_1 v a_1] \\ &= \sum_{sa_1, b_1} \sum_{v, r \in L(k)} r^* (v^{-1}) a_1^* r (*b_1) [b_1 v a_1]\end{aligned}$$

Therefore

$$\rho(\Delta'_k) = \sum_{sa_1, b_1} \sum_{v, r \in L(k)} r^* (v^{-1}) [a_1^* r (*b_1)] [b_1 v a_1]$$

Let $a \in_k T$, $w \in L(k)$ and $b \in T_k$ be fixed. Then:

$$X_{[a^*w(*b)]}(\rho(\Delta'_k)) = \sum_{s, v \in L(k)} w^* (v^{-1}) [bva]$$

Note that

$$\begin{aligned}Y_{[a^*w(*b)]}(\mu_k P) &= \rho^{-1} \left(X_{[a^*w(*b)]}(\rho(\mu_k P)) \right) \\ &= \rho^{-1} \left(X_{[a^*w(*b)]}(\rho(\Delta'_k)) \right) \\ &= \rho^{-1} \left(X_{[a^*w(*b)]}(\rho(\Delta'_k)) \right)\end{aligned}$$

Therefore

$$Y_{[a^*w(*b)]}(\mu_k P) = \sum_{s, v \in L(k)} w^* (v^{-1}) [bva] = c_k \sum_{v \in L(k)} w^* (v^{-1}) [bva]$$

Consequently

$$\begin{aligned}\pi_{bt} \bar{\gamma} \xi_{sa}(n) &= \sum_{w \in L(k)} (t^{-1})^* (sw) Y_{[a^*w(*b)]}(\mu_k P) n \\ &= c_k \sum_{v, w \in L(k)} (t^{-1})^* (sw) w^* (v^{-1}) [bva] n \\ &= c_k \sum_{v \in L(k)} (t^{-1})^* \left(s \sum_{w \in L(k)} w^* (v^{-1}) w \right) [bva] n \\ &= c_k \sum_{v \in L(k)} (t^{-1})^* (sv^{-1}) [bva] n\end{aligned}$$

By 2.3, $(t^{-1})^* (sv^{-1}) = v^* (ts)$. Then

$$\begin{aligned}
\pi_{bt}\overline{\gamma}\xi_{sa}(n) &= c_k \sum_{v \in L(k)} v^*(ts)[bva]n \\
&= c_k \left[b \sum_{v \in L(k)} v^*(ts)va \right] n \\
&= c_k [btsa]n \\
&= c_k btsan \\
&= c_k \pi_{bt}\beta\alpha\xi_{sa}(n)
\end{aligned}$$

As a direct consequence of 5.4, 5.5, 5.6, 5.8 and 5.9 we conclude that

$$\begin{aligned}
(5.10) \quad \ker(\overline{\alpha}) &= \text{im}(\beta), \text{im}(\overline{\alpha}) = \frac{\ker(\gamma)}{\text{im}(\beta)} \oplus \text{im}(\gamma) \oplus \{0\} \oplus \{0\}, \\
\ker(\overline{\beta}) &= \frac{\ker(\gamma)}{\text{im}(\beta)} \oplus \{0\} \oplus \{0\} \oplus V_k, \text{im}(\overline{\beta}) = \ker(\alpha), \\
\ker(\overline{\gamma}) &= \ker(\beta\alpha), \text{im}(\overline{\gamma}) = \text{im}(\beta\alpha).
\end{aligned}$$

Therefore

$$\overline{\overline{V}}_k = \frac{\ker(\overline{\beta})}{\ker(\beta) \cap \text{im}(\overline{\alpha})} = V_k$$

and hence $\overline{\overline{V}} = V$.

On the other hand

$$\overline{\overline{N}}_k = \frac{\ker(\overline{\gamma})}{\text{im}(\beta)} \oplus \text{im}(\overline{\gamma}) \oplus \frac{\ker(\overline{\alpha})}{\text{im}(\overline{\gamma})} \oplus \overline{V}_k$$

and by 5.10

$$\overline{\overline{N}}_k = \frac{\ker(\beta\alpha)}{\ker(\alpha)} \oplus \text{im}(\beta\alpha) \oplus \frac{\text{im}(\beta)}{\text{im}(\beta\alpha)} \oplus \frac{\ker(\beta)}{\ker(\beta) \cap \text{im}(\alpha)}$$

We now make the following observations:

- (a) the map α induces an isomorphism $\ker(\beta\alpha)/\ker(\alpha) \xrightarrow{\theta_1} \ker(\beta) \cap \text{im}(\alpha)$;
- (b) the map β induces an isomorphism $\text{im}(\alpha)/(\ker(\beta) \cap \text{im}(\alpha)) \xrightarrow{\theta_2} \text{im}(\beta\alpha)$;
- (c) the map β induces an isomorphism $N_k/(\ker(\beta) + \text{im}(\alpha)) \xrightarrow{\theta_3} \text{im}(\beta)/\text{im}(\beta\alpha)$.
- (d) there exists an isomorphism $\ker(\beta)/(\ker(\beta) \cap \text{im}(\alpha)) \xrightarrow{\theta_4} (\ker(\beta) + \text{im}(\alpha))/\text{im}(\alpha)$.

Using these isomorphisms, we can identify $\overline{\overline{N}}_k$ with the space

$$\overline{\overline{N}}_k \xrightarrow{f} (\ker(\beta) \cap \text{im}(\alpha)) \oplus \frac{\text{im}(\alpha)}{\ker(\beta) \cap \text{im}(\alpha)} \oplus \frac{N_k}{\ker(\beta) + \text{im}(\alpha)} \oplus \frac{\ker(\beta) + \text{im}(\alpha)}{\text{im}(\alpha)}$$

via the map

$$f = \begin{bmatrix} \theta_1 & 0 & 0 & 0 \\ 0 & \theta_2^{-1} & 0 & 0 \\ 0 & 0 & \theta_3^{-1} & 0 \\ 0 & 0 & 0 & \theta_4 \end{bmatrix}$$

We have canonical projections

$$\begin{aligned}\bar{\pi}_1 : \ker(\beta\alpha) &\rightarrow \frac{\ker(\beta\alpha)}{\ker(\alpha)} \\ \bar{\pi}_2 : \operatorname{im}(\beta) &\rightarrow \frac{\operatorname{im}(\beta)}{\operatorname{im}(\beta\alpha)} \\ \bar{\pi}_3 : \operatorname{im}(\alpha) &\rightarrow \frac{\operatorname{im}(\alpha)}{\ker(\beta) \cap \operatorname{im}(\alpha)} \\ \bar{\pi}_4 : N_k &\rightarrow \frac{N_k}{\ker(\beta) + \operatorname{im}(\alpha)} \\ \bar{\pi}_5 : \ker(\beta) + \operatorname{im}(\alpha) &\rightarrow \frac{\ker(\beta) + \operatorname{im}(\alpha)}{\operatorname{im}(\alpha)}\end{aligned}$$

and inclusion maps

$$\begin{aligned}\bar{i}_1 : \operatorname{im}(\bar{\gamma}) &\rightarrow N_{out} \\ \bar{i}_2 : \ker(\beta\alpha) &\rightarrow N_{in} \\ \bar{i}_3 : \operatorname{im}(\beta\alpha) &\rightarrow N_k \\ \bar{j} : \ker(\bar{\alpha}) &\rightarrow N_{out}\end{aligned}$$

We now choose splitting data as follows:

- (a) Let $\bar{p} : N_{in} \rightarrow \ker(\beta\alpha)$ be a map of left D_k -modules such that $\bar{p}\bar{i}_2 = id_{\ker(\beta\alpha)}$.
- (b) Let $\bar{\sigma} : \operatorname{im}(\beta)/\operatorname{im}(\beta\alpha) \rightarrow \operatorname{im}(\beta)$ be a map of left D_k -modules such that $\bar{\pi}_2\bar{\sigma} = id_{\operatorname{im}(\beta)/\operatorname{im}(\beta\alpha)}$.

For each $a \in_k T$, there exists an F -linear map

$$\bar{\bar{N}}(a) : N_{\tau(a)} \rightarrow (\ker(\beta) \cap \operatorname{im}(\alpha)) \oplus \frac{\operatorname{im}(\alpha)}{\ker(\beta) \cap \operatorname{im}(\alpha)} \oplus \frac{N_k}{\ker(\beta) + \operatorname{im}(\alpha)} \oplus \frac{\ker(\beta) + \operatorname{im}(\alpha)}{\operatorname{im}(\alpha)}$$

such that

$$\begin{aligned}\bar{\Pi}_1 \bar{\bar{N}}(a) &= -\theta_1 \bar{\pi}_1 \bar{p} \xi_{e_k a} \\ \bar{\Pi}_2 \bar{\bar{N}}(a) &= -\theta_2^{-1} \bar{\gamma}' \xi_{e_k} a \\ \bar{\Pi}_3 \bar{\bar{N}}(a) &= \bar{\Pi}_4 \bar{\bar{N}}(a) = 0\end{aligned}$$

Let $n \in N_{\tau(a)}$. Then

$$\begin{aligned}-\theta_1 \bar{\pi}_1 \bar{p} \xi_{e_k a}(n) &= -\theta_1 (\bar{p} \xi_{e_k a}(n) + \ker(\alpha)) = -\alpha \bar{p} \xi_{e_k a}(n) \\ -\theta_2^{-1} (\bar{\gamma}' (\xi_{e_k a}(n))) &= -\theta_2^{-1} (c_k \beta \alpha \xi_{e_k a}(n)) = -c_k \bar{\pi}_3 \alpha \xi_{e_k a}(n)\end{aligned}$$

Thus the map $\bar{\bar{N}}(a)$ takes the following form:

$$\begin{aligned}\bar{\Pi}_1 \bar{\bar{N}}(a) &= -\alpha \bar{p} \xi_{e_k a} \\ \bar{\Pi}_2 \bar{\bar{N}}(a) &= -c_k \bar{\pi}_3 \alpha \xi_{e_k a} \\ \bar{\Pi}_3 \bar{\bar{N}}(a) &= \bar{\Pi}_4 \bar{\bar{N}}(a) = 0\end{aligned}\tag{5.11}$$

Similarly, for each $b \in T_k$, there exists an F -linear map

$$\overline{\overline{N}}(b) : (\ker(\beta) \cap \operatorname{im}(\alpha)) \oplus \frac{\operatorname{im}(\alpha)}{\ker(\beta) \cap \operatorname{im}(\alpha)} \oplus \frac{N_k}{\ker(\beta) + \operatorname{im}(\alpha)} \oplus \frac{\ker(\beta) + \operatorname{im}(\alpha)}{\operatorname{im}(\alpha)} \longrightarrow N_{\sigma(b)}$$

given by

$$\begin{aligned}\overline{\overline{N}}(b)\overline{J}_1 &= \overline{\overline{N}}(b)\overline{J}_4 = 0 \\ \overline{\overline{N}}(b)\overline{J}_2 &= c_k^{-1}\pi_{be_k}\overline{i}_3\theta_2 \\ \overline{\overline{N}}(b)\overline{J}_3 &= c_k^{-1}\pi_{be_k}\overline{j}\overline{\sigma}\theta_3\end{aligned}$$

To complete the proof of Theorem 2, it remains to construct an isomorphism of F -vector spaces $\psi : \overline{\overline{N}}_k \rightarrow N_k$ such that

$$\begin{aligned}\psi\overline{\overline{N}}(a) &= \alpha\xi_{e_ka} \\ \pi_{be_k}\beta\psi &= \overline{\overline{N}}(b)\end{aligned}$$

for every $a \in_k T$ and $b \in T_k$. Choose some sections

$$\begin{aligned}\sigma_1 &: \operatorname{im} \alpha / (\ker \beta \cap \operatorname{im} \alpha) \rightarrow \operatorname{im} \alpha \\ \sigma_2 &: (\ker \beta + \operatorname{im} \alpha) / \operatorname{im}(\alpha) \rightarrow \ker \beta + \operatorname{im} \alpha \\ \sigma_3 &: N_k / (\ker \beta + \operatorname{im} \alpha) \rightarrow N_k\end{aligned}$$

so that they satisfy

$$\operatorname{im}(\sigma_1) = \alpha(\ker \overline{p}), \operatorname{im}(\sigma_2) \subseteq \ker(\beta), \operatorname{im}(\beta\sigma_3) = \operatorname{im}(\overline{\sigma})$$

Define $\psi : \overline{\overline{N}}_k \rightarrow N_k$ as

$$\psi = \begin{bmatrix} -i_1 & -c_k^{-1}i_2\sigma_1 & -c_k^{-1}\sigma_3 & -i_3\sigma_2 \end{bmatrix}$$

where

$$\begin{aligned}i_1 &: \ker(\beta) \cap \operatorname{im}(\alpha) \rightarrow N_k \\ i_2 &: \operatorname{im}(\alpha) \rightarrow N_k \\ i_3 &: \ker(\beta) + \operatorname{im}(\alpha) \rightarrow N_k\end{aligned}$$

are the inclusion maps. We now show that

$$(5.12) \quad \psi\overline{\overline{N}}(a) = \alpha\xi_{e_ka}$$

Since $\overline{p} : N_{in} \rightarrow \ker(\beta\alpha)$ is a retraction, then $N_{in} = \ker(\overline{p}) \oplus \ker(\beta\alpha)$. On the other hand, since $\ker(\alpha) \subseteq \ker(\beta\alpha)$ then there exists a submodule $L \subseteq \ker(\beta\alpha)$ such that $\ker(\alpha) \oplus L = \ker(\beta\alpha)$. Therefore $N_{in} = \ker(\overline{p}) \oplus \ker(\alpha) \oplus L$.

Let $n \in N_{\tau(a)}$, then $\xi_{e_ka}(n) \in N_{in}$. Thus, $\xi_{e_ka}(n) = n_1 + n_2 + n_3$ for some uniquely determined $n_1 \in \ker(\overline{p})$, $n_2 \in \ker(\alpha)$ and $n_3 \in L$. Therefore:

$$\alpha\xi_{e_ka}(n) = \alpha(n_1 + n_2 + n_3) = \alpha(n_1 + n_3)$$

On the other hand:

$$\begin{aligned}
 (\psi \overline{N}(a))(n) &= \psi(-\alpha \overline{p} \xi_{e_k a}(n), -c_k \overline{\pi}_3 \alpha \xi_{e_k a}(n), 0, 0) \\
 &= \alpha \overline{p}(n_1 + n_2 + n_3) + c_k^{-1} c_k \sigma_1 \overline{\pi}_3 \alpha(n_1 + n_3) \\
 &= \alpha \overline{p}(n_2 + n_3) + \alpha(n_1) \\
 &= \alpha(n_2 + n_3) + \alpha(n_1) \\
 &= \alpha(n_1 + n_3) \\
 &= \alpha \xi_{e_k a}(n)
 \end{aligned}$$

and the proof of 5.12 is now complete.

We now verify that

$$(5.13) \quad \pi_{be_k} \beta \psi = \overline{N}(b)$$

Let $n_1 \in \ker(\beta) \cap \text{im}(\alpha)$, $n_2 \in \text{im}(\alpha)$, $n_3 \in N_k$, $n_4 \in \ker(\beta) + \text{im}(\alpha)$. Then

$$\begin{aligned}
 \overline{N}(b)(n_1, \overline{\pi}_3(n_2), \overline{\pi}_4(n_3), \overline{\pi}_5(n_4)) &= c_k^{-1} \pi_{be_k} \overline{i}_3 \theta_2 \overline{\pi}_3(n_2) + c_k^{-1} \pi_{be_k} \overline{j} \overline{\sigma} \theta_3 \overline{\pi}_4(n_3) \\
 &= c_k^{-1} b \cdot n_2 + c_k^{-1} b \cdot n_3 \\
 &= c_k^{-1} \psi_0^{-1}(b) n_2 + c_k^{-1} \psi_0^{-1}(b) n_3 \\
 &= -c_k^{-1} b n_2 - c_k^{-1} b n_3
 \end{aligned}$$

On the other hand

$$\pi_{be_k} \beta \psi(n) = \pi_{be_k} \beta \left(-n_1 - c_k^{-1} i_2 \sigma_1 \overline{\pi}_3(n_2) - c_k^{-1} \sigma_3 \overline{\pi}_4(n_3) - i_3 \sigma_2 \overline{\pi}_5(n_4) \right)$$

Since $n_1 \in \ker(\beta)$ and $\text{im}(\sigma_2) \subseteq \ker(\beta)$ we see that the above expression is equal to

$$\begin{aligned}
 \pi_{be_k} \beta (-c_k^{-1} i_2 \sigma_1 \overline{\pi}_3(n_2) - c_k^{-1} \sigma_3 \overline{\pi}_4(n_3)) &= \pi_{be_k} \beta (-c_k^{-1} n_2 - c_k^{-1} n_3) \\
 &= -c_k^{-1} b n_2 - c_k^{-1} b n_3
 \end{aligned}$$

and 5.13 follows. Finally, we extend ψ to an isomorphism of F -vector spaces $\psi' : \overline{N} \rightarrow N$ defined as the identity map on $\bigoplus_{i \neq k} \overline{N}_i$.

□

6. NEARLY MORITA EQUIVALENCE

Throughout this section, $\mathcal{F}_S(M)/R(P)\text{-mod}_k$ denotes the category of finite dimensional left $\mathcal{F}_S(M)/R(P)$ -modules modulo the ideal of morphisms which factor through direct sums of the left $\mathcal{F}_S(M)$ -simple module at k .

In this section we prove that the categories $\mathcal{F}_S(M)/R(P)\text{-mod}_k$ and $\mathcal{F}_S(\overline{\mu}_k M)/R(\overline{\mu}_k P)\text{-mod}_k$ are equivalent, where $(\mathcal{F}_S(\overline{\mu}_k M), \mathcal{F}_S(\overline{\mu}_k P))$ denotes the mutation at k in the sense of [2, p.56].

For each finite dimensional left $\mathcal{F}_S(M)/R(P)$ -module, we choose splitting data $(p_N, (\sigma_2)_N)$. Let $u : N \rightarrow N^1$ be a morphism of left $\mathcal{F}_S(M)/R(P)$ -modules. Then u induces D_k -linear maps:

$$\begin{aligned} u_{out} &: N_{out} \rightarrow N_{out}^1 \\ u_{in} &: N_{in} \rightarrow N_{in}^1 \end{aligned}$$

such that for each $a, a_1 \in_k T, r, r_1 \in L(k)$

$$\pi_{r_1 a_1}^1 u_{in} \xi_{ra} = \delta_{r_1 a_1, ra} u$$

and for each $b, b_1 \in T_k, r, r_1 \in L(k)$

$$\pi_{b_1 r_1}^1 u_{out} \xi_{br} = \delta_{b_1 r_1, br} u$$

We also have D_k -linear maps

$$\begin{aligned} \alpha &: N_{in} \rightarrow N_k; \alpha_1 : N_{in}^1 \rightarrow N_k^1 \\ \beta &: N_k \rightarrow N_{out}; \beta_1 : N_k^1 \rightarrow N_{out}^1 \\ \gamma &: N_{out} \rightarrow N_{in}; \gamma_1 : N_{out}^1 \rightarrow N_{in}^1 \end{aligned}$$

Then we have the following equalities

$$\begin{aligned} u_k \alpha &= \alpha_1 u_{in}; u_{out} \beta = \beta_1 u_k \\ u_{in} \gamma &= \gamma_1 u_{out} \end{aligned}$$

The map u_{out} induces D_k -linear maps

$$\begin{aligned} u_1 &: \ker(\gamma) \rightarrow \ker(\gamma_1) \\ u_4 &: \operatorname{im}(\beta) \rightarrow \operatorname{im}(\beta_1) \end{aligned}$$

The map u_{in} induces D_k -linear maps

$$\begin{aligned} u_2 &: \operatorname{im}(\gamma) \rightarrow \operatorname{im}(\gamma_1) \\ u_3 &: \ker(\alpha) \rightarrow \ker(\alpha_1) \end{aligned}$$

The map u_1 induces a D_k -linear map

$$\underline{u}_1 : \ker(\gamma) / \operatorname{im}(\beta) \rightarrow \ker(\gamma_1) / \operatorname{im}(\beta_1)$$

likewise, the map u_3 induces a D_k -linear map

$$\underline{u}_3 : \ker(\alpha) / \operatorname{im}(\gamma) \rightarrow \ker(\alpha_1) / \operatorname{im}(\gamma_1)$$

so we have a D_k -linear map

$$h = \underline{u}_1 \oplus u_2 \oplus \underline{u}_3 : \overline{N}_k \rightarrow \overline{N}_k^1$$

Then we define a linear map of left S -modules:

$$\overline{u} : \overline{N} \rightarrow \overline{N}^1$$

as $\overline{u}_i = u_i$ for every $i \neq k$ and $\overline{u}_k = h$.

Definition 14. Following [7] we say that $u \in \operatorname{Hom}_S({}_S L_1, {}_S L_2)$ is confined to k if $u_i : e_i L_1 \rightarrow e_i L_2$ is the zero map for all $i \neq k$. Note that a map of left $\mathcal{F}_S(M)$ -modules $u : L_1 \rightarrow L_2$ factors through a direct sum of the left $\mathcal{F}_S(M)$ -simple module at k if and only if u is confined to k .

Lemma 10. *Let $u : N \rightarrow N^1$ be a map of finite dimensional left $\mathcal{F}_S(M)/R(P)$ -modules. Then there exists a map of left S -modules $\overline{\rho(u)} : \overline{N} \rightarrow \overline{N}^1$, confined to k , and such that $\overline{u} + \overline{\rho(u)}$ is a map of left $\mathcal{F}_S(\mu_k M)$ -modules.*

Proof. Let $(p = p_N, \sigma_2 = (\sigma_2)_N)$ and $(p_1, \sigma_{2,1})$ be the splitting data for N and N^1 , respectively. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{im}(\gamma) & \longrightarrow & \ker(\alpha) & \xrightarrow{\pi_2} & \ker(\alpha)/\text{im}(\gamma) \longrightarrow 0 \\
 & & \downarrow u_2 & & \downarrow u_3 & & \downarrow \underline{u}_3 \\
 0 & \longrightarrow & \text{im}(\gamma_1) & \longrightarrow & \ker(\alpha_1) & \xrightarrow{\pi_2^1} & \ker(\alpha_1)/\text{im}(\gamma_1) \longrightarrow 0
 \end{array}$$

thus $\pi_2^1(u_3\sigma_2 - \sigma_{2,1}\underline{u}_3) = \underline{u}_3\pi_2\sigma_2 - \underline{u}_3 = \underline{u}_3 - \underline{u}_3 = 0$. Hence there exists a linear map of left S -modules:

$$\rho_1 = u_3\sigma_2 - \sigma_{2,1}\underline{u}_3 : \ker(\alpha)/\text{im}(\gamma) \rightarrow \text{im}(\gamma_1).$$

Similarly, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker(\gamma) & \xrightarrow{j} & N_{out} & \xrightarrow{\gamma} & \text{im}(\gamma) \longrightarrow 0 \\
 & & \downarrow u_1 & & \downarrow u_{out} & & \downarrow u_2 \\
 0 & \longrightarrow & \ker(\gamma_1) & \xrightarrow{j_1} & N_{out^1} & \xrightarrow{\gamma_1} & \text{im}(\gamma_1) \longrightarrow 0
 \end{array}$$

hence $(u_1p - p_1u_{out})j = u_1pj - p_1u_{out}j = u_1 - p_1j_1u_1 = u_1 - u_1 = 0$.

Therefore, there exists a linear map of left S -modules:

$$\rho_2 : \text{im}(\gamma) \rightarrow \ker(\gamma_1)$$

such that $\rho_2\gamma = p_1u_{out} - u_1p$.

Define $\overline{\rho(u)} : \overline{N} \rightarrow \overline{N^1}$ as follows: $\overline{\rho(u)}_i = 0$ for all $i \neq k$ and $\overline{\rho(u)}_k : \overline{N}_k \rightarrow \overline{N^1}_k$ is the following linear map of left S -modules:

$$\begin{bmatrix} 0 & \pi_1^1\rho_2 & 0 \\ 0 & 0 & \rho_1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let us show that the map $\overline{u} + \overline{\rho(u)}$ is in fact a map of left $\mathcal{F}_S(\mu_k M)$ -modules.

Let $n \in N_{\sigma(b)}$. On one hand

$$\begin{aligned}
 (\overline{u} + \overline{\rho(u)}) (\overline{N}(*b)(n)) &= (-\underline{u}_1\pi_1p\xi_{be_k}(n) - \pi_1^1\rho_2\gamma\xi_{be_k}(n), -u_2\gamma\xi_{be_k}(n), 0) \\
 &= (-\pi_1^1u_1p\xi_{be_k}(n) - \pi_1^1\rho_2\gamma\xi_{be_k}(n), -u_2\gamma\xi_{be_k}(n), 0)
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 \overline{N^1}(*b)(u_{\sigma(b)}(n)) &= (-\pi_1^1p_1\xi_{be_k}(u_{\sigma(b)}(n)), -\gamma_1\xi_{be_k}(u_{\sigma(b)}(n)), 0) \\
 &= (-\pi_1^1p_1u_{out}(\xi_{be_k}(n)), -\gamma_1u_{out}(\xi_{be_k}(n)), 0) \\
 &= (-\pi_1^1u_1p\xi_{be_k}(n) - \pi_1^1\rho_2\gamma\xi_{be_k}(n), -u_2\gamma\xi_{be_k}(n), 0)
 \end{aligned}$$

Therefore

$$(\overline{u} + \overline{\rho(u)}) \overline{N}(*b) = \overline{N^1}(*b)u_{\sigma(b)}$$

Now for each $a \in T_k$, $x \in \ker(\gamma)/\text{im}(\gamma)$, $y \in \text{im}(\gamma)$ and $z \in \ker(\alpha)/\text{im}(\gamma)$ we have

$$\begin{aligned} u_{\tau(a)} \overline{N}(a^*)(x, y, z) &= c_k^{-1} u_{\tau(a)} (\pi_{e_k a} i(y) + \pi_{e_k a} j'_1 \sigma_2(z)) \\ &= c_k^{-1} (\pi_{e_k a}^1 i_1 u_2(y) + \pi_{e_k a}^1 j'_1 u_3 \sigma_2(z)) \end{aligned}$$

On the other hand

$$\begin{aligned} \overline{N^1}(a^*) (\overline{u} + \overline{\rho(u)}) (x, y, z) &= \overline{N^1}(a^*) \left((\overline{u}_1(x), u_2(y), \overline{u}_3(z)) + (\pi_1^1 \rho_2(y), \rho_1(z), 0) \right) \\ &= c_k^{-1} \left(\pi_{e_k a}^1 i_1 u_2(y) + \pi_{e_k a}^1 j'_1 \sigma_{2,1} \underline{u}_3(z) + \pi_{e_k a}^1 i_1 \rho_1(z) \right) \\ &= c_k^{-1} \left(\pi_{e_k a}^1 i_1 u_2(y) + \pi_{e_k a}^1 j'_1 (\sigma_{2,1} \underline{u}_3(z) + \rho_1(z)) \right) \\ &= c_k^{-1} \left(\pi_{e_k a}^1 i_1 u_2(y) + \pi_{e_k a}^1 j'_1 u_3 \sigma_2(z) \right) \end{aligned}$$

thus $u_{\tau(a)} \overline{N}(a^*) = \overline{N^1}(a^*) (\overline{u} + \overline{\rho(u)})$, as was to be shown. \square

Proposition 8. *There exists a faithful functor $\tilde{\mu}_k : \mathcal{F}_S(M)/R(P) - \text{mod}_k \rightarrow \mathcal{F}_S(\mu_k M)/R(\mu_k P) - \text{mod}_k$*

Proof. First we define a functor $G : \mathcal{F}_S(M)/R(P) - \text{mod} \rightarrow \mathcal{F}_S(\mu_k M)/R(\mu_k P) - \text{mod}_k$ as $G(N) = \overline{N}$ and given a map of left $\mathcal{F}_S(M)/R(P)$ -modules $u : N \rightarrow N^1$, we define:

$$G(u) = \overline{u} + \overline{\rho(u)} : \overline{N} \rightarrow \overline{N^1}$$

On the other hand, if $v : N^1 \rightarrow N^2$ is a map of left $\mathcal{F}_S(M)/R(P)$ -modules then $G(vu) = \overline{vu} + \overline{\rho(vu)}$. Since ρ is concentrated in k and $\overline{vu} = \overline{v} \overline{u}$ one sees that $G(vu) = G(v)G(u)$ so that G preserves composition. Since $\overline{\rho(id_N)} = 0$ then $G(id_N) = id_{\overline{N}}$ so that G is indeed a covariant (additive) functor.

Finally, note that $G(u) = 0$ if and only if $\overline{u} + \overline{\rho(u)}$ is confined to k , which happens if and only if \overline{u} is confined to k and the latter happens if only if u is confined to k , as was to be shown. \square

Let $\varphi : \mathcal{F}_S(M) \rightarrow \mathcal{F}_S(M_1)$ be an algebra isomorphism such that $\varphi|_S = id_S$. Let P be a potential in $\mathcal{F}_S(M)$.

Throughout the rest of this section, $J(M, P)$ will denote the quotient algebra $\mathcal{F}_S(M)/R(P)$.

The isomorphism φ induces a functor

$$H_\varphi : J(M, P) - \text{mod} \rightarrow J(M_1, \varphi(P)) - \text{mod}$$

given as follows. In objects, $H_\varphi(N) = {}^\varphi N$; that is, $H_\varphi(N) = N$ as S -left modules, and given $n \in N$ and $z \in \mathcal{F}_S(M_1)$, $z \cdot n = \varphi^{-1}(z)n$. Clearly, $\text{Hom}_{J(M, P)}(N, N^1) = \text{Hom}_{J(M_1, \varphi(P))}({}^\varphi N, {}^\varphi N^1)$. Therefore, we let $H_\varphi(u) = u$. This gives an equivalence of categories

$$(6.1) \quad H_\varphi : J(M, P) - \text{mod} \rightarrow J(M_1, \varphi(P)) - \text{mod}$$

Now suppose that $M = M_1 \oplus M_2$, and $Q = Q_1 + W$ where Q_1 is a reduced potential in $\mathcal{F}_S(M_1)$ and W is a trivial potential in $\mathcal{F}_S(M_2)$. Then the restriction functor

$$(6.2) \quad \text{res} : J(M, Q) - \text{mod} \rightarrow J(M_1, Q_1) - \text{mod}$$

yields also an equivalence of categories.

On the other hand, by [2, Theorem 8.17] there exists a right-equivalence

$$\psi : \mathcal{F}_S(\mu_k^2 M) \rightarrow \mathcal{F}_S(M \oplus M')$$

such that $\psi(\mu_k^2 P)$ is cyclically equivalent to $P + W$ where W is a trivial potential in $\mathcal{F}_S(M')$. Thus, using 6.1 and 6.2 we obtain equivalence of categories:

$$(6.3) \quad H_\psi : J(\mu_k^2 M, \mu_k^2 P) - \text{mod} \rightarrow J(M \oplus M', P + W) - \text{mod}$$

$$(6.4) \quad \text{res} : J(M \oplus M', P + W) - \text{mod} \rightarrow J(M, P) - \text{mod}$$

composing the above functors yields an equivalence of categories

$$(6.5) \quad G(\psi) = \text{res}H_\psi : J(\mu_k^2 M, \mu_k^2 P) - \text{mod} \rightarrow J(M, P) - \text{mod}$$

In what follows, given $N \in J(M, P) - \text{mod}$, we will denote by ${}_S N$ the S -left module underlying N . In particular, ${}_S G(\psi)(N) = {}_S N$.

Lemma 11. *Let \mathcal{A}, \mathcal{B} be F -categories and let $C : \mathcal{A} \rightarrow \mathcal{B}$, $D : \mathcal{B} \rightarrow \mathcal{A}$ be F -functors such that D is faithful and there exists a natural isomorphism $\text{id}_{\mathcal{A}} \cong DC$. Then C is fully faithful. Moreover, if D is full, then $\text{id}_{\mathcal{B}} \cong CD$.*

Proof. For each $X \in \text{Ob}(\mathcal{A})$ there exists an isomorphism

$$\phi_X : X \rightarrow DC(X)$$

Now let $u \in \text{Hom}_{\mathcal{A}}(X, X_1)$. By naturality, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_X} & DC(X) \\ \downarrow u & & \downarrow DC(u) \\ X_1 & \xrightarrow{\phi_{X_1}} & DC(X_1) \end{array}$$

Thus $\phi_{X_1} u = DC(u) \phi_X$. Therefore if $C(u) = 0$, then $u = 0$. This shows that C is faithful. Now let $h \in \text{Hom}_{\mathcal{B}}(C(X), C(X_1))$. Let

$$\lambda = \phi_{X_1}^{-1} D(h) \phi_X : X \rightarrow X_1$$

Since the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\phi_X} & DC(X) \\ \downarrow \lambda & & \downarrow DC(\lambda) \\ X_1 & \xrightarrow{\phi_{X_1}} & DC(X_1) \end{array}$$

then $\lambda = \phi_{X_1}^{-1}DC(\lambda)\phi_X$. Therefore $D(h) = DC(\lambda)$ and thus $h = C(\lambda)$. It follows that C is full. Now suppose that D is full, then for each $Y \in \text{Ob}(\mathcal{B})$ there exists an isomorphism $\phi_{D(Y)} : D(Y) \rightarrow DCD(Y)$. Since D is full, then $\phi_{D(Y)} = D(\psi_Y)$ for some $\psi_Y \in \text{Hom}_{\mathcal{B}}(Y, CD(Y))$. Let us show that ψ_Y is natural. Let $f \in \text{Hom}_{\mathcal{B}}(Y_1, Y_2)$. We have to prove the following diagram is commutative

$$(6.6) \quad \begin{array}{ccc} Y & \xrightarrow{\psi_Y} & CD(Y) \\ \downarrow f & & \downarrow CD(f) \\ Y_1 & \xrightarrow{\psi_{Y_1}} & CD(Y_1) \end{array}$$

Consider the following commutative diagram

$$\begin{array}{ccc} D(Y) & \xrightarrow{\phi_{D(Y)}} & DCD(Y) \\ \downarrow D(f) & & \downarrow DCD(f) \\ D(Y_1) & \xrightarrow{\phi_{D(Y_1)}} & DCD(Y_1) \end{array}$$

thus $DCD(f)\phi_{D(Y)} = \phi_{D(Y_1)}D(f)$. This implies that $DCD(f)D(\psi_Y) = D(\psi_{Y_1})D(f)$. Because D is faithful it follows that $CD(f)\psi_Y = \psi_{Y_1}f$ and 6.6 commutes, as was to be shown. \square

By 6.5 there exists an equivalence of categories

$$G(\psi) = \text{res}H_\psi : J(\mu_k^2 M, \mu_k^2 P) - \text{mod} \rightarrow J(M, P) - \text{mod}$$

and this functor descends to a functor $G(\psi)_k$ in the quotient category $J(M, P) - \text{mod}_k$ which is the category $J(M, P) - \text{mod}$, modulo the ideal of morphisms which factor through direct sums of the simple module at k . Thus, we have a functor

$$G(\psi)_k : J(\mu_k^2 M, \mu_k^2 P) - \text{mod}_k \rightarrow J(M, P) - \text{mod}_k$$

Proposition 9. *There exists a natural isomorphism of functors $\text{id}_{J(M, P) - \text{mod}_k} \cong G(\psi)_k \tilde{\mu}_k^2$.*

Proof. Let $u \in \text{Hom}_{J(M, P) - \text{mod}_k}(N, N^1)$. Remembering the proof of Proposition 8 we have

$$\begin{aligned} \tilde{\mu}_k(u) &= \overline{u} + \overline{\rho(u)} = u_1 : \overline{N} \rightarrow \overline{N^1} \\ \tilde{\mu}_k^2(u) &= \tilde{\mu}_k(u_1) = \overline{u_1} + \overline{\rho(u_1)} = u_2 : \overline{\overline{N}} \rightarrow \overline{\overline{N^1}} \end{aligned}$$

Using the notation introduced in the proof of Theorem 2, we have isomorphisms of $J(\mu_k^2 M, \mu_k^2 P)$ -left modules

$$\begin{aligned} \psi' &: G(\psi)_k \tilde{\mu}_k^2 N \rightarrow N \\ \psi'_1 &: G(\psi)_k \tilde{\mu}_k^2 N^1 \rightarrow N^1 \end{aligned}$$

It remains to show that the following diagram commutes in $J(M, P) - \text{mod}_k$.

$$(6.7) \quad \begin{array}{ccc} G(\psi)_k \tilde{\mu}_k^2 N & \xrightarrow{\psi'} & N \\ \downarrow u_2 & & \downarrow u \\ G(\psi)_k \tilde{\mu}_k^2 N & \xrightarrow{\psi'_1} & N^1 \end{array}$$

but this is true since $u\psi' - \psi'_1 u_2$ is confined to k . This completes the proof. \square

Proposition 10. *There exists a natural isomorphism of functors*

$$\tilde{\mu}_k G(\psi)_k \tilde{\mu}_k \cong id_{J(\mu_k M, \mu_k P) - \text{mod}_k}.$$

Proof. By Proposition 8, the functor

$$\tilde{\mu}_k : J(\mu_k M, \mu_k P) - \text{mod}_k \rightarrow J(\mu_k^2 M, \mu_k^2 P) - \text{mod}_k$$

is faithful, hence $G(\psi)_k \tilde{\mu}_k$ is faithful as well. By Lemma 11 and Proposition 8, the functor $\tilde{\mu}_k$ is fully faithful. Therefore, $G(\psi)_k \tilde{\mu}_k$ is full. The result now follows by applying Lemma 11. \square

Theorem 3. *Let P be a potential in $\mathcal{F}_S(M)$. If $\mu_k P$ is splittable, then there exists an equivalence of categories:*

$$\mu_k : J(M, P) - \text{mod}_k \rightarrow J(\bar{\mu}_k M, \bar{\mu}_k P) - \text{mod}_k$$

Proof. Since $\mu_k P$ is splittable, then by [2, Theorem 7.15] there exists an algebra isomorphism $\varphi : \mathcal{F}_S(\mu_k M) \rightarrow \mathcal{F}_S(\bar{\mu}_k M \oplus M')$, with $\varphi|_S = id_S$, and such that $\varphi(\mu_k P)$ is cyclically equivalent to $\bar{\mu}_k P + W$ where W is a trivial potential in $\mathcal{F}_S(M')$. By 6.1 and 6.2 there exists an equivalence of categories

$$G(\varphi) : J(\mu_k M, \mu_k P) - \text{mod} \rightarrow J(\bar{\mu}_k M, \bar{\mu}_k P) - \text{mod}$$

which induces an equivalence of categories

$$G(\varphi)_k : J(\mu_k M, \mu_k P) - \text{mod}_k \rightarrow J(\bar{\mu}_k M, \bar{\mu}_k P) - \text{mod}_k$$

By Propositions 9 and 10, the categories $J(M, P) - \text{mod}_k$ and $J(\mu_k M, \mu_k P) - \text{mod}_k$ are equivalent; hence, we get an equivalence of categories

$$\mu_k : J(M, P) - \text{mod}_k \rightarrow J(\bar{\mu}_k M, \bar{\mu}_k P) - \text{mod}_k$$

as desired. \square

REFERENCES

- [1] M. Auslander, I. Reiten and S. Smalø, *Representation theory of Artin algebras*. Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, Cambridge, (1997).
- [2] R. Bautista and D. López-Aguayo, *Potentials for some tensor algebras*. arXiv:1506.05880.
- [3] R. Bautista, L. Salmerón and R. Zuazua, *Differential tensor algebras and their module categories*. London Mathematical Society Lecture Note Series, 362. Cambridge University Press, Cambridge, (2009).
- [4] A.B. Buan, O. Iyama, I. Reiten and D. Smith, *Mutation of cluster-tilting objects and potentials*. American Journal of Mathematics, 133 (2011), no. 4, 835-887. arXiv:0804.3813.
- [5] L. Demonet, *Mutations of group species with potentials and their representations. Applications to cluster algebras*. arXiv:1003.5078.
- [6] H. Derksen, J. Weyman, A. Zelevinsky. *Quivers with potentials and their representations I: Mutations*. Selecta Math. **14** (2008), no. 1, 59-119. arXiv:0704.0649.
- [7] H. Derksen, J. Weyman, A. Zelevinsky. *Quivers with potentials and their representations II: Applications to cluster algebras*. J. Amer. Math. Soc. **23** (2010), no. 3, 749-790. arXiv:0904.0676.

- [8] D. Labardini-Fragoso, A. Zelevinsky. *Strongly primitive species with potentials I: Mutations*. Boletín de la Sociedad Matemática Mexicana (Third series), Vol. **22**, (2016), Issue 1, 47-115. <http://arxiv.org/pdf/1306.3495v1.pdf>

CENTRO DE CIENCIAS MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, APARTADO POSTAL 61-3 (XANGARI), MORELIA, MICHOACÁN, MEXICO 58089

E-mail address: raymundo@matmor.unam.mx

E-mail address: dlopez@matmor.unam.mx